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On the Torelli theorem for Deligne-Hitchin moduli spaces

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Abstract

We prove a Torelli theorem for the parabolic Deligne-Hitchin moduli space, and compare it with previous Torelli theorems for non-parabolic Deligne-Hitchin moduli spaces.

1 Introduction

Let X be a smooth projective complex curve. The classical Torelli theorem says that we can recover the isomorphism class of the curve from the isomorphism class of the Jacobian $J(X)$ (which we think of as the moduli space of degree 0 line bundles on X) with the standard polarization $\Theta(X)$. In other words, if $(J(X), \Theta(X))$ is isomorphic to $(J(X'), \Theta(X'))$ as polarized varieties, then there is an isomorphism between X and X'. This has been generalized for other moduli spaces. A theorem which asserts that we can recover the curve X from the isomorphism class of some moduli space is called a Torelli theorem.

For instance, a Torelli theorem is known for the moduli space of semistable vector bundles with fixed determinant on X [Tj, NR, MN, KP]. For the moduli space of Higgs bundles, the Torelli theorem was proved in [BG]. In the case of the moduli space of parabolic bundles and parabolic Higgs bundles we recover the curve with the parabolic points [BBB, BHK, GL, BGL].

Deligne [De] has given a glueing construction of the twistor space of the moduli space of Higgs bundles [Hi2, §9], and this complex analytic space is called the Deligne-Hitchin moduli space (see [Si3] for the description).

In [BGHL] a Torelli theorem for the Deligne-Hitchin moduli space is proved (it should be noted that in [BGHL] we fix the determinant of the underlying vector bundle to be trivial, so the structure group is $\mathrm{SL}(r, \mathbb{C})$, but in [Si3] only the degree is fixed, so the structure group is $\mathrm{GL}(r, \mathbb{C})$).

In [BGH], Deligne glueing is used to construct a Deligne-Hitchin moduli space for a semisimple structure group and a Torelli type theorem is proved for it.

In this talk we report on the work [AG], where we prove a Torelli theorem for the Deligne-Hitchin moduli space for parabolic vector bundles, which is constructed using Deligne glueing. The detailed constructions and proofs (sometimes with more generality) can be found in the original paper [AG]. The aim here is to highlight the main ideas and to explain the similarities and differences with [BGHL] and [BGH].

The theorem we are going to prove is (see next section for the definition of the Deligne-Hitchin moduli space)

Theorem 1.1. *Fix a genus $g \geq 3$, a number of points $n \geq 1$, and a rank $r \geq 2$. Let X and X' be smooth complex projective curves of genus g, and $D \subset X$, $D' \subset X'$ divisors consisting of n distinct points. Let α and α' be systems of weights satisfying the conditions in Remark 2.1. Assume $\sum_{x\in D} \beta(x)$ and $\sum_{x'\in D'} \beta'(x')$ are coprime to r.*

If $r = 2$ and the parabolic Deligne-Hitchin moduli spaces are isomorphic as analytic varieties

$$\mathcal{M}_{\mathrm{DH}}(X, r, \alpha) \cong \mathcal{M}_{\mathrm{DH}}(X', r, \alpha')$$

then (X', D') is isomorphic to either (X, D) or $(\overline{X}, D)$.

We remark that the condition $r = 2$ is imposed because we use [BBB]. The situation is the following: the moduli spaces of parabolic Higgs bundles

$$\mathcal{M}(X, r, \alpha, \xi) \quad \text{and} \quad \mathcal{M}(\overline{X}, r, -\alpha, \overline{\xi})$$

with $\xi = \mathcal{O}_X(-\sum_{x\in D} \beta(x)\cdot x)$ embed into the parabolic Deligne-Hitchin moduli space. We show (for arbitrary r) that we can recover the images of these embeddings just from the isomorphism class of the parabolic Deligne-Hitchin moduli space. Then we apply the Torelli theorem for parabolic vector bundles [BBB] (which requires $r = 2$) to obtain our Torelli theorem. Therefore, if the result of [BBB] were generalized to arbitrary r, we would automatically get a Torelli theorem for parabolic Deligne-Hitchin moduli spaces for arbitrary rank r.

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2 Parabolic Deligne-Hitchin moduli space

In this section we recall some basic definitions and the constructions of the moduli spaces involved. Let X be a smooth projective complex curve, and let D be a divisor consisting of $n \geq 1$ distinct points (these are called the parabolic points). A parabolic vector bundle on X is a holomorphic vector bundle E or rank r endowed with a weighted filtration of the fiber E_x over each parabolic point $x \in D$:

$$E_x = E_{x,1} \supsetneq E_{x,2} \supsetneq \cdots \supsetneq E_{x,l_x} \supsetneq E_{x,l_x+1} = 0$$

$$0 \leq \alpha_1(x) < \alpha_2(x) < \cdots < \alpha_{l_x}(x) < 1 \; .$$

We say that $\alpha_i(x)$ is the weight associated to $E_{x,i}$. In this article we will assume that the filtrations are full flags, i.e., $l_x = r$ for all parabolic points x.

Let $(E, E_\bullet)$ be a parabolic vector bundle. We define its parabolic degree to be

$$\text{pardeg}(E, E_\bullet) = \deg(E) + \sum_{x\in D}\sum_{i=1}^{r} \alpha_i(x) \; . \tag{2.1}$$

Let $F \subset E$ be a subbundle. There is a parabolic structure induced on F as follows. For each parabolic point $x \in X$ we obtain a filtration $F_{x,i}$ of F_x by looking at the intersections $F_x \cap E_{x,j}$ for all j. The weight $\beta_i(x)$ of $F_{x,i}$ is defined as

$$\beta_i(x) = \max_j\{\alpha_j(x) : F_x \cap E_{x,j} = F_{x,i}\} \; .$$

We say that a parabolic bundle $(E, E_\bullet)$ is stable (respectively, semistable) if for all proper parabolic subbundles $F \subsetneq E$, with the induced parabolic structure,

$$\text{pardeg}(F, F_\bullet) < \text{pardeg}(E, E_\bullet) \qquad \text{(respectively, } \leq) \; .$$

Let $\mathcal{M}(X, r, \alpha, \xi)$ (or $\mathcal{M}(r, \alpha, \xi)$, or just $\mathcal{M}$, if the rest of the data is clear from the context) be the moduli space of semistable parabolic vector bundles on X of rank r, weights α and fixed determinant isomorphic to the line bundle ξ. It is projective of dimension

$$\dim \mathcal{M}(X, r, \alpha, \xi) = (r^2 - 1)(g - 1) + \frac{n(r^2 - r)}{2} \; .$$

Remark 2.1. We will assume the following conditions throughout the article

1. The weights are generic (see [AG] for the explicit meaning of generic in our situation), and the filtrations are full flags.

2. The weights are "concentrated", meaning that $\alpha_r(x) - \alpha_1(x) < \frac{4}{nr^2}$ for all parabolic points $x \in D$.

3. For each parabolic point the sum of the weights is an integer, i.e., for all parabolic points $x \in D$

$$\beta(x) := \sum_{i=1}^{r} \alpha_i(x) \in \mathbb{Z} \; .$$

4. The degree of ξ (the determinant of the vector bundles E) and the rank r are coprime.

Proposition 2.2. *Under the above assumptions, there are no strictly semistable parabolic vector bundles with respect the the weights* α*, the moduli space*

$$\mathcal{M}(X,r,\alpha,\xi)$$

is smooth, projective and parameterizes stable parabolic bundles. Furthermore, if $(E,E_\bullet)$ *is a parabolic vector bundle, then it is stable as a vector bundle if and only if it is stable as a parabolic vector bundle.*

For the proof, see [AG]. Recall that, as the rank and degree are coprime, no vector bundle is strictly semistable as a vector bundle.

A strongly parabolic Higgs bundle is a parabolic vector bundle $(E,E_\bullet)$ together with a homomorphism, called Higgs field,

$$\Phi : E \longrightarrow E \otimes K(D)$$

such that, for all parabolic points $x \in D$, the homomorphism induced in the fiber satisfies

$$\Phi(E_{x,i}) \subset E_{x,i+1} \otimes K(D)|_x$$

where K is the canonical line bundle of the curve X. A weakly parabolic Higgs bundle is defined analogously, but requiring the weaker condition

$$\Phi(E_{x,i}) \subset E_{x,i} \otimes K(D)|_x \ .$$

Unless otherwise stated, all Higgs fields will be strongly parabolic.

A subbundle $F \subset E$ is called Φ-invariant if $\Phi(F) \subset F \otimes K(D)$. We say that a parabolic Higgs bundle $(E,E_\bullet,\Phi)$ is stable (respectively, semistable) if the inequality (2.1) holds for all proper Φ-invariant subbundles.

Let $\mathcal{M}_{\text{Higgs}}(X,r,\alpha,\xi)$ (or just $\mathcal{M}_{\text{Higgs}}$ if the rest of the data is clear from the context) be the moduli space of semistable parabolic Higgs bundles on X of rank r, weights α, fixed determinant isomorphic to the line bundle ξ, and $\operatorname{tr}\Phi = 0$. Recall that we are assuming that the weights are generic. This implies that there are no strictly semistable parabolic Higgs bundles, and that this moduli space is smooth. Its dimension is

$$\dim \mathcal{M}_{\text{Higgs}}(X,r,\alpha,\xi) = 2(r^2-1)(g-1) + n(r^2-r) \ .$$

The cotangent space of $\mathcal{M}(r,\alpha,\xi)$ sits inside this moduli space as an open subset

$$T^*\mathcal{M}(X,r,\alpha,\xi) \subset \mathcal{M}_{\text{Higgs}}(X,r,\alpha,\xi) \ . \tag{2.2}$$

This open subset consists of the parabolic Higgs bundles whose underlying parabolic bundle is stable. The complement consists of parabolic Higgs bundles which are stable as a parabolic Higgs bundle, but whose underlying parabolic bundle is unstable.

To see (2.2), note that the tangent space to a point $(E, E_\bullet)$ is

$$H^1(X, \mathrm{ParEnd}^0(E, E_\bullet)) \ ,$$

the first cohomology of the sheaf of traceless weakly parabolic endomorphisms (the traceless condition comes from the fact that the determinant is fixed). By Serre duality

$$H^1(X, \mathrm{ParEnd}^0(E, E_\bullet))^* \cong H^0(X, \mathrm{SParEnd}^0(E, E_\bullet) \otimes K(D)) \ ,$$

the space of strongly parabolic Higgs fields with $\operatorname{tr} \Phi = 0$, and any such Higgs bundle is stable because the underlying parabolic vector bundle is stable. This explains why it is natural to require that the Higgs field is traceless and strongly parabolic (as opposed to weakly parabolic).

The moduli space of parabolic Higgs bundles is endowed with a $\mathbb{C}^*$-action which, for each $t \in \mathbb{C}^*$, sends $(E, E_\bullet, \Phi)$ to $(E, E_\bullet, t\Phi)$. This is compatible, using (2.2), with scalar multiplication on the cotangent bundle of $\mathcal{M}(X, r, \alpha, \xi)$.

We now recall the definition of the Hitchin map and spectral curves in this setting. Denote by S the total space of the line bundle $K(D)$, let

$$p : S = \underline{\mathrm{Spec}}\, \mathrm{Sym}^\bullet(K(D))^* \longrightarrow X$$

be the projection and let $x \in H^0(S, p^*K(D))$ be the tautological section. Taking the characteristic polynomial of a Higgs field

$$\det(x \cdot \mathrm{id} - p^*\Phi) = x^r + \widetilde{s}_1 x^{r-1} + \widetilde{s}_2 x^{r-2} + \cdots + \widetilde{s}_r$$

we obtain sections $\widetilde{s}_i \in H^0(S, p^*K^iD^i)$ and it can be shown that these come from sections on X, i.e., there are sections $s_i \in H^0(X, K^iD^i)$ such that $p^*s_i = \widetilde{s}_i$.

Recall that we are assuming that Φ is strongly parabolic. Then the residue at each parabolic point is nilpotent and hence the eigenvalues of Φ vanish at the divisor D. Therefore, for each i the section s_i belongs to the subspace $H^0(X, K^iD^{i-1})$, where we use the shorthand $K^iD^j = K^i \otimes \mathcal{O}_X(D)^j$.

Furthermore, we are assuming that the trace of Φ is identically zero, so $s_1 = 0$. We then define the Hitchin space as

$$\mathcal{H}_0 = H^0(X, K^2D) \oplus \cdots \oplus H^0(X, K^rD^{r-1}) \ . \tag{2.3}$$

The Hitchin map is defined by taking the characteristic polynomial of the Higgs field, i.e., to each Higgs field Φ we associate the point in the Hitchin space defined by the elements s_i, $2 \leq i \leq r$ defined above

$$H : \mathcal{M}_{\mathrm{Higgs}}(r, \alpha, \xi) \longrightarrow \mathcal{H}_0 \ .$$

This map is projective with connected fibers.

From now on we will assume

$$\xi = \mathcal{O}_X\Big(- \sum_{x\in D} \beta(x) \cdot x\Big)$$

so that the parabolic degree of our vector bundles will be zero (recall that $\beta(x)$ is an integer because of the assumptions in Remark 2.1).

Deligne introduced the notion of λ-connection ($\lambda \in \mathbb{C}$) which interpolates between Higgs bundles and usual connections: if $\lambda = 0$ we get a Higgs bundle, and if $\lambda = 1$ we get a usual connection.

Definition 2.3. Let $(E, E_\bullet)$ be a parabolic bundle with an isomorphism $\det(E) \cong \xi$. A λ-connection, for the group $\mathrm{SL}(r, \mathbb{C})$, on X is a triple $(E, E_\bullet, \nabla)$ where ∇ is a $\mathbb{C}$-linear homomorphism

$$\nabla : E \longrightarrow E \otimes K(D)$$

such that

1. (Leibniz) If f is a holomorphic function and s is a holomorphic section of E (both over some open set of X),
$$\nabla(fs) = f\nabla(s) + \lambda s \otimes df \ .$$
2. For each parabolic point $x \in D$ the residue satisfies
$$\mathrm{Res}_x(\nabla)(E_{x,i}) \subseteq E_{x,i} \ .$$
3. For each parabolic point $x \in D$, the action of the residue of ∇ on $E_{x,i}/E_{x,i+1}$ is multiplication by $\lambda\alpha_i(x)$ (this is well defined because the residue preserves the filtration by the previous property).
4. The operator $\mathrm{tr}(\nabla) : \det(E) \longrightarrow \det(E) \otimes K(D)$ induced by ∇ coincides with $\lambda\nabla_{\xi,\beta}$ (defined below).

A parabolic vector bundle $(E, E_\bullet)$ induces a parabolic structure on the determinant $\xi = \det(E)$, with weights $\beta(x) = \sum_{i=1}^r \alpha_i(x)$, and using the correspondence between parabolic bundles of parabolic degree 0 and connections, this gives a connection on ξ (with poles along D) which we denote $\nabla_{\xi,\beta}$.

A subbundle $F \subset E$ is called ∇-invariant if $\nabla(F) \subset F \otimes K(D)$. We say that a λ-connection is stable (respectively, semistable) if the inequality (2.1) holds for all proper ∇-invariant parabolic subbundles.

Let $\mathcal{M}_{\mathrm{Hodge}}(X, r, \alpha, \xi)$ (or just $\mathcal{M}_{\mathrm{Hodge}}$ if the rest of the data is clear from the context) be the moduli space of tuples $(\lambda, E, E_\bullet, \nabla)$ where ∇ is a semistable λ-connection (these objects are called parabolic Hodge bundles). It has a projection

$$\mathrm{pr}_\lambda : \mathcal{M}_{\mathrm{Hodge}}(X, r, \alpha, \xi) \longrightarrow \mathbb{C} \ . \qquad (2.4)$$

Note that, if $\lambda = 0$, then a λ-connection is just a Higgs bundle, so we have

$$\mathrm{pr}_\lambda^{-1}(0) = \mathcal{M}_{\mathrm{Higgs}}(X, r, \alpha, \xi) \ .$$

The Hodge moduli space is endowed with a $\mathbb{C}^*$-action which, for each $t \in \mathbb{C}^*$, sends

$$(\lambda, E, E_\bullet, \nabla) \mapsto (t\lambda, E, E_\bullet, t\nabla) \ . \tag{2.5}$$

This extends the standard $\mathbb{C}^*$-action on the moduli space of parabolic Higgs bundles, and the projection pr_λ is equivariant, using the standard scalar multiplication on $\mathbb{C}$.

On the other hand, if $\lambda = 1$, then a λ-connection is a holomorphic flat connection on a parabolic vector bundle, i.e., a logarithmic connection such that the residue over each parabolic point x acts on $E_{x,i}/E_{x,i+1}$ as $\alpha_i(x)$. Therefore, the fiber over $\lambda = 1$ is the moduli space of parabolic $\mathrm{SL}(r, \mathbb{C})$ connections

$$\mathrm{pr}_\lambda^{-1}(1) = \mathcal{M}_{\mathrm{conn}}(X, r, \alpha, \xi) \ .$$

If ∇ is a λ-connection with $\lambda \neq 0$, then $(1/\lambda)\nabla$ is a usual connection, so the fiber of pr_λ over any $\lambda \neq 0$ is isomorphic to the moduli space of parabolic connections (i.e., meromorphic connections which respect the parabolic filtration). Therefore, the Hodge moduli space shows that the moduli space of Higgs bundles is a degeneration of the moduli space of connections (or, equivalently, the moduli space of connections is a deformation of the moduli space of Higgs bundles).

Now we are going to describe Deligne's glueing in the parabolic setting. Let $X_\mathbb{R}$ be the real manifold underlying X with the orientation induced by the complex structure. Fix a base point x_0 and a positively oriented simple loop

$$\gamma_x \in \pi_1(X_\mathbb{R} \setminus D, x_0)$$

around each parabolic point $x \in D$. Let $\mathcal{M}_{\mathrm{rep}}(X_\mathbb{R}, r, \alpha)$ be the set of irreducible representations

$$\rho : \pi_1(X_\mathbb{R} \setminus D, x_0) \longrightarrow \mathrm{SL}(r, \mathbb{C})$$

of the fundamental group, up to conjugation by $\mathrm{SL}(r, \mathbb{C})$, such that, for each parabolic point $x \in D$, the eigenvalues of $\rho(\gamma_x)$ are $\{e^{-2\pi i \alpha_i(x)}\}$. Note that, since the sum of the weights for each parabolic point is an integer, the determinant of $\rho(\gamma_x)$ is 1.

Simpson [Si1] has given a Riemann-Hilbert correspondence between stable parabolic connections of parabolic degree 0 and stable filtered local systems of degree 0 for the general linear group $\mathrm{GL}(n, \mathbb{C})$. It has a version for our setting (the group is $\mathrm{SL}(r, \mathbb{C})$ and the eigenvalues of the monodromies around the parabolic points are fixed by the weights α as above) which gives a biholomorphism

$$\mathrm{RH}_X : \mathcal{M}_{\mathrm{rep}}(X_\mathbb{R}, r, \alpha) \xrightarrow{\cong} \mathcal{M}_{\mathrm{conn}}(X, r, \alpha, \xi) \ . \tag{2.6}$$

Using this biholomorphism and the action (2.5) we obtain a holomorphic open embedding

$$\begin{array}{rcl} \mathbb{C}^* \times \mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}}, r, \alpha) & \hookrightarrow & \mathcal{M}_{\mathrm{Hodge}}(X, r, \alpha, \xi) \\ (t, \rho) & \mapsto & (t, t \cdot \mathrm{RH}_X(\rho)) \end{array} \tag{2.7}$$

onto the open locus $\mathrm{pr}_\lambda^{-1}(\mathbb{C}^*) \subset \mathcal{M}_{\mathrm{Hodge}}(X, r, \alpha, \xi)$.

Let $J_{\mathbb{R}}$ be the almost complex structure on $X_{\mathbb{R}}$ coming from X. Then $-J_{\mathbb{R}}$ is also an almost complex structure on $X_{\mathbb{R}}$ whose corresponding Riemann surface will be denoted by $\overline{X}$. Let $\overline{\xi}$ be the line bundle on $\overline{X}$ given by the complex structure conjugate to ξ. As a topological line bundle, it is isomorphic to ξ^{-1}. Note that the underlying manifold $\overline{X}_{\mathbb{R}}$ to $\overline{X}$ is the same as $X_{\mathbb{R}}$ but the induced orientation is the opposite. Therefore, if we consider the same loops γ_x defined above, we can identify

$$\mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}}, r, \alpha) = \mathcal{M}_{\mathrm{rep}}(\overline{X}_{\mathbb{R}}, r, -\alpha) .$$

We will also use the Riemann-Hilbert isomorphism for $\overline{X}$

$$\mathrm{RH}_{\overline{X}} : \mathcal{M}_{\mathrm{rep}}(\overline{X}_{\mathbb{R}}, r, -\alpha) \xrightarrow{\cong} \mathcal{M}_{\mathrm{conn}}(\overline{X}, r, -\alpha, \overline{\xi}) .$$

The parabolic Deligne-Hitchin moduli space is defined by glueing

$$\mathcal{M}_{\mathrm{DH}}(X, r, \alpha) := \mathcal{M}_{\mathrm{Hodge}}(X, r, \alpha, \xi) \cup \mathcal{M}_{\mathrm{Hodge}}(\overline{X}, r, -\alpha, \overline{\xi})$$

along the image of $\mathbb{C}^* \times \mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}}, r, \alpha) = \mathbb{C}^* \times \mathcal{M}_{\mathrm{rep}}(\overline{X}_{\mathbb{R}}, r, -\alpha)$ using $\mathrm{RH}_{\overline{X}}$ and RH_X, identifying

$$(\lambda, \lambda \cdot \mathrm{RH}_X(\rho)) \in \mathcal{M}_{\mathrm{Hodge}}(X, r, \alpha, \xi)$$

with

$$(\lambda^{-1}, \lambda^{-1} \cdot \mathrm{RH}_{\overline{X}}(\rho)) \in \mathcal{M}_{\mathrm{Hodge}}(\overline{X}, r, -\alpha, \overline{\xi}) .$$

The projection pr_λ in (2.4) extends to a morphism to $\mathbb{P}^1$ which we denote by the same letter

$$\mathrm{pr}_\lambda : \mathcal{M}_{\mathrm{DH}}(X, r, \alpha) \longrightarrow \mathbb{P}^1 .$$

This moduli space will be denoted $\mathcal{M}_{\mathrm{DH}}$ if the rest of the data is clear from the context. It is clear, staring at the definition, that there is a holomorphic isomorphism,

$$\mathcal{M}_{\mathrm{DH}}(X, r, \alpha) \cong \mathcal{M}_{\mathrm{DH}}(\overline{X}, r, -\alpha) \tag{2.8}$$

covering the antipodal map on $\mathbb{P}^1$. Because of this isomorphism, we cannot expect to recover the isomorphism class of (X, D), but only the unordered pair $\{(X, D), (\overline{X}, D)\}$.

3 Torelli theorem

Taking the Higgs field to be zero, we have an embedding of the moduli space of parabolic vector bundles in the moduli space of parabolic Higgs bundles. Furthermore, a Higgs bundle is the same thing as a 0-connection, so we also have an embedding into the Hodge moduli space. Finally, the Hodge moduli space is an open subset of the Deligne-Hitchin moduli space, so we have an embedding

$$\mathcal{M}(X,r,\alpha,\xi) \subset \mathcal{M}_{\mathrm{DH}}(X,r,\alpha) \tag{3.1}$$

Note that the image of this embedding sits on the fiber over $\lambda = 0$. By definition, $\mathcal{M}_{\mathrm{Hodge}}(\overline{X}, r, -\alpha, \overline{\xi})$ is also an open subset of $\mathcal{M}_{\mathrm{DH}}(X,r,\alpha)$, so we also have an embedding

$$\mathcal{M}(\overline{X}, r, -\alpha, \overline{\xi}) \subset \mathcal{M}_{\mathrm{DH}}(X,r,\alpha) \tag{3.2}$$

and the image of this embedding sits over $\lambda = \infty \in \mathbb{P}^1$.

The strategy of the proof is the following. Given the isomorphism class of $\mathcal{M}_{\mathrm{DH}}(X,r,\alpha)$ as an analytic variety, we want to identify the image of $\mathcal{M}(X,r,\alpha,\xi)$ under the first embedding. Then we have the isomorphism class of $\mathcal{M}(X,r,\alpha,\xi)$, and we can apply the Torelli theorem [BBB] for the moduli space of parabolic bundles, in order to recover the isomorphism class of the pointed curve. Unfortunately, the Torelli theorem for parabolic bundles is only known for $r = 2$, and this is why we have to restrict our attention to rank 2.

Because of the isomorphism (2.8), we will not be able to distinguish between the images of (3.1) and (3.2).

The idea of the proof is to look at all vector fields on $\mathcal{M}_{\mathrm{DH}}(X,r,\alpha)$, i.e., sections of its tangent bundle. We look at the analytic subset of this moduli space defined as the simultaneous zeroes of all vector fields. More precisely, we will prove the following

Proposition 3.1. *Let Z be an irreducible component of*

$$\{z \in \mathcal{M}_{\mathrm{DH}}(X,r,\alpha) : \eta(z) = 0 \quad \text{for all } \eta \in H^0(\mathcal{M}_{\mathrm{DH}}, T\mathcal{M}_{\mathrm{DH}})\}$$

such that

$$\dim Z = \dim \mathcal{M}(X,r,\alpha,\xi) \ .$$

Then Z is the image of one of the embeddings (3.1) *and* (3.2).

We start by showing that, at most, these two subsets satisfy the condition:

Proposition 3.2. *Let Z be an irreducible component of $\mathcal{M}_{\mathrm{DH}}^{\mathbb{C}^*}$, the fixed locus of the standard $\mathbb{C}^*$-action. Then*

$$\dim Z \leq \dim \mathcal{M}(X,r,\alpha,\xi)$$

with equality only for Z the image of the embedding (3.1) *or* (3.2).

Proof. Here we use the standard $\mathbb{C}^*$-action. This produces a vector field which vanishes at the fixed point locus. The projection pr_λ is equivariant with respect to the standard $\mathbb{C}^*$-action, so the irreducible components of the fixed point locus of the $\mathbb{C}^*$-action have to sit in the fibers of λ equal to 0 or ∞. We will first consider the case $\lambda = 0$, so the fixed locus sits in the moduli space of parabolic Higgs bundles. The Hitchin map is equivariant with respect to the $\mathbb{C}^*$-action on the parabolic Higgs moduli, so the fixed locus has to sit in $H^{-1}(0)$, which is called the nilpotent cone.

The nilpotent cone is equidimensional of dimension $\dim \mathcal{M}(X, r, \alpha, \xi)$. Exactly one of its components is the moduli space of parabolic bundles $\mathcal{M}(X, r, \alpha, \xi)$, and this is a fixed locus of the standard $\mathbb{C}^*$-action. The standard $\mathbb{C}^*$-action is nontrivial on the other components (by an argument similar to [Si2, Lemma 11.9]). We remark that the other fixed locus subvarieties correspond to variations of Hodge structures.

The case $\lambda = \infty$ is the same, but looking at the moduli space of parabolic Higgs bundles on $\overline{X}$. □

In [BGHL] the same proof was used. In [BGH] we do not have at our disposal an analogue of [Si2, Lemma 11.9], and the proof is based on a study of the infinitesimal deformations of the points fixed by the standard $\mathbb{C}^*$-action.

Now we show the opposite direction, i.e., we show that the images of these embeddings do satisfy the condition. We would like to show that all global vector fields vanish at $\mathcal{M}$, and we will do this by showing the stronger condition

$$H^0(\mathcal{M}(X, r, \alpha, \xi), T\mathcal{M}_{\mathrm{DH}}(X, r, \alpha)|_{\mathcal{M}(X,r,\alpha,\xi)}) = 0 \ .$$

This will be proved in several steps.

Lemma 3.3. *The holomorphic cotangent bundle*

$$T^*\mathcal{M}(X, r, \alpha, \xi) \longrightarrow \mathcal{M}(X, r, \alpha, \xi)$$

does not admit any nonzero holomorphic section.

Proof. Since we are assuming full flags, it is known that $\mathcal{M}(X, r, \alpha, \xi)$ is rational [BY, Theorem 6.1], and it is well-known that a smooth rational projective variety does not admit nonzero holomorphic 1-forms. □

In [BGHL] this is Lemma 2.1, and it was proved using the Hecke transform. The point is that, in the setting of [BGHL], we are looking at differentials (1-forms) on the moduli space of vector bundles with trivial determinant, which is not smooth. By doing a Hecke transform, we relate it to the moduli space of vector bundles with fixed determinant $\mathcal{O}_X(x_0)$ for a fixed point x_0. This is a smooth unirational projective variety [Se, p. 53], so it does not admit any nonzero holomorphic 1-form. An argument using the Hecke correspondence shows that the same holds for the moduli space of vector bundles with trivial determinant.

In [BGH] this is Proposition 4.2, and it was proved using "abelianization". By fixing a spectral curve, we obtained a dominant rational map from an abelian variety to the moduli space of bundles. Using this map, a holomorphic 1-form ω on the moduli space would give a rational 1-form on the abelian variety which, by a codimension argument, is shown to be defined on the whole abelian variety. Any 1-form on an abelian variety is closed, so ω is closed, but the first cohomology of the moduli space is zero [AB, Ch. 10], so $\omega = df$ for a holomorphic function on the moduli space, but since this is projective, f is constant, so $\omega = 0$.

Lemma 3.4. *The holomorphic tangent bundle*

$$T\mathcal{M}(X, r, \alpha, \xi) \longrightarrow \mathcal{M}(X, r, \alpha, \xi)$$

does not admit any nonzero holomorphic section

Proof. We adapt an argument of Hitchin [Hi2, Theorem 6.2]. A section s of the tangent bundle gives, by contraction, a function on the total space of the cotangent bundle

$$s^\sharp : T^*\mathcal{M}(X, r, \alpha, \xi) \longrightarrow \mathbb{C}$$

which is linear on the fibers. The total space of the cotangent bundle is an open subset of the moduli of parabolic Higgs bundles, and the codimension of the complement is greater than two (here we use a calculation of Faltings [F, Lemma II.6 and V.(iii), p. 561]), so it extends to a function on the moduli of parabolic Higgs bundles. This descends, using the Hitchin map, to a function on the Hitchin space, which is homogeneous of degree 1 with respect to the standard $\mathbb{C}^*$-action (because $s^\sharp$ is linear on the fibers). But the Hitchin space (2.3) has no linear part, so this function has to be zero, and hence $s = 0$. □

In [BGHL] this is Lemma 2.2, and the proof is the same. In [BGH] this is Proposition 4.1, and we use a result of Faltings [F, Corollary III.3].

Corollary 3.5. *The tangent bundle to the parabolic Higgs moduli space, restricted to the moduli space of parabolic bundles, has no nonzero section, i.e.*

$$H^0(\mathcal{M}, T\mathcal{M}_{\mathrm{Higgs}}|_{\mathcal{M}}) = 0 \; .$$

Proof. There is a short exact sequence on $\mathcal{M}$

$$0 \longrightarrow T\mathcal{M} \longrightarrow T\mathcal{M}_{\mathrm{Higgs}}|_{\mathcal{M}} \longrightarrow N \longrightarrow 0$$

where N is the normal of the embedding of $\mathcal{M}$ in $\mathcal{M}_{\mathrm{Higgs}}$, but $N \cong T^*\mathcal{M}$ because we have a closed embedding (the zero section) and an open embedding

$$\mathcal{M} \subset T^*\mathcal{M} \subset \mathcal{M}_{\mathrm{Higgs}} \; .$$

Now we use Lemmas 3.3 and 3.4.

□

Let $\mathcal{M}^{\text{s-par}}_{\text{Hodge}}$ and $\mathcal{M}^{\text{s-par}}_{\text{conn}}$ be the subsets of $\mathcal{M}_{\text{Hodge}}$ and $\mathcal{M}_{\text{conn}}$ where the underlying parabolic bundle is stable as a parabolic bundle. Recall that we are assuming generic weights, so that $\mathcal{M}^{\text{s-par}} = \mathcal{M}$.

Proposition 3.6. *The projection*

$$\text{pr}_E : \mathcal{M}^{\text{s-par}}_{\text{conn}} \longrightarrow \mathcal{M}$$

sending a connection to the underlying parabolic bundle, admits no holomorphic section.

Proof. Let s be such a section. The parabolic version of the Riemann-Hilbert correspondence (2.6) gives a biholomorphism between the moduli space of stable parabolic connections and the moduli space of irreducible representations of the fundamental group of the open curve, and this moduli space is an affine variety. Then the section s would give a holomorphic map from the projective variety $\mathcal{M}$ to an affine variety, but this map has to be constant, so the map s does not exist. □

In [BGHL] this is Proposition 3.2. There we remark that the projection pr_E is a torsor under the cotangent bundle. A section of pr_E then gives a nonzero cohomology class in the first cohomology class of the cotangent bundle, but in the setting of [BGHL] it is proved that this cohomology group is zero.

In [BGH] this is Proposition 4.4. We remark that the projection pr_E is isomorphic to the torsor of holomorphic connections on certain determinant line bundle, but this line bundle is ample, so it does not admit holomorphic connections.

Corollary 3.7. *Let*

$$\text{pr}_E : \mathcal{M}^{\text{s-par}}_{\text{Hodge}} \longrightarrow \mathcal{M}$$

be the projection which sends each Hodge bundle to the underlying parabolic bundle. The only section s of this projection is the standard embedding of $\mathcal{M}$ in $\mathcal{M}_{\text{Hodge}}$.

Proof. The composition of the section s with the projection pr_λ is a regular function on the projective variety $\mathcal{M}$, hence it is constant which, after scaling with the $\mathbb{C}^*$-action, we may assume it is 1 or 0.

It cannot be 1, because the fiber of pr_λ over 1 is the moduli space of connections, and it would contradict Proposition 3.6.

If it is 0 then, since the fiber of pr_λ over 0 is the moduli space of parabolic Higgs bundles, the section s factors through $\mathcal{M}^{\text{s-par}}_{\text{Higgs}}$, but this is isomorphic to the total space of $T^*\mathcal{M}$. Then s is a section of this vector bundle and, by Lemma 3.3, it has to be the zero section. □

Corollary 3.8.

$$H^0(\mathcal{M}(X, r, \alpha, \xi), T\mathcal{M}_{\text{Hodge}}(X, r, \alpha, \xi)|_{\mathcal{M}(X,r,\alpha,\xi)}) = 0$$

Proof. Using the short exact sequence on $\mathcal{M}$

$$0 \longrightarrow T\mathcal{M} \longrightarrow T\mathcal{M}^{\text{s-par}}_{\text{Hodge}}|_{\mathcal{M}} \longrightarrow N \longrightarrow 0$$

and Lemma 3.4, it suffices to show that N has no nonzero sections. Note that N is isomorphic to $\mathcal{M}^{\text{s-par}}_{\text{Hodge}}$ as varieties over $\mathcal{M}$, sending $(\lambda, E, E_\bullet, \nabla)$ to the derivative at $t = 0$ of the map $\mathbb{C} \longrightarrow \mathcal{M}^{\text{s-par}}_{\text{Hodge}}$ given by

$$t \mapsto (t\lambda, E, E_\bullet, t\nabla)$$

so by Corollary 3.7 we conclude that N has no nonzero section. □

We are now ready to prove the main Theorem 1.1.

Corollary 3.8 implies that the images of the embeddings (3.1) and (3.2) are in the fixed point locus of any $\mathbb{C}^*$-action on the parabolic Deligne-Hitchin moduli space (if they were not, the derivative of the action would give vector field whose restriction to these images is nonzero). This together with Proposition 3.2, shows that these are the only irreducible components Z with this property having

$$\dim Z = \dim \mathcal{M}$$

hence proving Proposition 3.1.

Finally, using Proposition 3.1, from the isomorphism class of the parabolic Deligne-Hitchin moduli space we recover the isomorphism class of $\mathcal{M}(X, r, \alpha, \xi)$ or $\mathcal{M}(\overline{X}, r, -\alpha, \bar{\xi})$. Therefore, assuming $r = 2$ we can apply [BBB] to finish the proof of the main Theorem.

The same ideas also give Torelli theorems for $\mathcal{M}_{\text{Higgs}}$ and $\mathcal{M}_{\text{Hodge}}$. Indeed, in these cases we again have an embedding of $\mathcal{M}$, and the image is characterized as the only irreducible component, with dimension $\dim \mathcal{M}$, of the simultaneous zeroes of the sections of the tangent bundle. See [AG] for details.

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