Strong and Covariant Morita equivalences in Deformation Quantization

by Henrique Bursztyn and Stefan Waldmann

Abstract

This note presents an overview of various aspects of the representation theory of star products, including different notions of module and Morita equivalence, as well as classification results. Along the way, we highlight many connections with the work of Nikolai Neumaier.

1   Introduction

A central theme in Nikolai Neumaier's work was formal deformation quantization [2] (see e.g. [41] for an introduction), a subject to which he gave many important contributions; within deformation quantization, the study of representations of star-product algebras was among his main topics of interest. This note presents an overview of various aspects of the representation theory of star products, including different notions of Morita equivalence as well as classification results, some of which had the direct influence of Nikolai's work.

Morita equivalence, in its original and most basic form, is an equivalence relation among unital rings which identifies those with equivalent "representation theories" (i.e., categories of left modules). The notion of Morita equivalence can be transferred to many other situations: basically, it can be formulated whenever one specifies a reasonable notion of representation of (or module over) a mathematical object. This note presents some instances of this idea when the mathematical object in question is a star-product algebra; as we will see, depending on the properties of star products that one wants to take into account, different notions of representation and Morita equivalence arise.

In order to find appropriate frameworks for star-product representations, it is convenient to recall the physical motivation of star products as models for observable algebras of quantum systems. Star products are formal associative deformations, in the sense of Gerstenhaber [18], of the commutative algebra of smooth, complex-valued functions $C^\infty(M)$ on a Poisson manifold (M, π), thought of as the classical phase space. A star product $\star$ makes the space of formal power series $C^\infty(M)[[\hbar]]$ (here $\hbar$ is viewed as a formal parameter) into a unital, associative algebra over the ring $\mathbb{C}[[\hbar]]$; a key requirement is that star products deform

the pointwise product of functions "in the direction" of the given Poisson structure π, meaning that the $\star$-commutator on $C^\infty(M)[[\hbar]]$ agrees (up to a constant), in first order, with the Poisson bracket on M.

The role of star-product algebras as observable algebras indicates that one should consider not only their ring structures, but also additional properties. In fact, a desirable scenario would be to use formal deformation quantization to eventually obtain C^*-algebras represented on Hilbert spaces. But this aim is hard to achieve: there are many technical difficulties in handling convergence issues for formal power series and finding C^*-norms for suitable classes of functions, although this can be done in specific examples. An alternative approach is to proceed within the framework of formal power series, observing that some properties of C^*-algebras and their representations carry over to the purely algebraic formal setting. Indeed, there are two important "C^*-like" features that one may consider for star products: first, by considering Hermitian star products, i.e. star products compatible with complex conjugation of functions (we assume the parameter $\hbar$ to be real),

$$\overline{f \star g} = \overline{g} \star \overline{f}, \qquad f, g \in C^\infty(M)[[\hbar]],$$

one endows star-product algebras with *-*involutions*; second, one may take into account notions of *positivity* (e.g. for algebra elements and linear functionals) resulting from the natural order structure on the ring $\mathbb{R}[[\hbar]]$ (a formal series $\sum_{r=0}^\infty \hbar^r a_r$ is declared to be *positive* if its first nonzero term is positive). These additional features of star products lead to notions of representations parallel to those for C^*-algebras [38, 39], and to an algebraic version of the concept of *strong Morita equivalence* [13]. On top of that, one may consider star products carrying symmetries, given by actions of a Hopf algebra H, and representations which are compatible with these symmetries. This leads to the notion of H-*covariant Morita equivalence*, studied by Nikolai in one of his last publications [21].

This note is organized as follows. Section 2 is divided in two parts: first, we review the usual classification of star products and their characteristic classes (highlighting Nikolai's contributions in this context) and, afterwards, we discuss the classification of star products with respect to ring-theoretic Morita equivalence. In Section 3 we consider algebras with additional properties and present various ways in which one can enhance the notions of (bi-)module and representation, by taking into account positivity and the presence of symmetries; these new (bi-)modules lead to refined notions of Morita equivalence, such as strong and covariant Morita equivalences, treated in Section 4. Here we emphasize the bicategorical approach to Morita equivalence: we describe different versions of Morita equivalence as isomorphisms in appropriate bicategories of bimodules with extra structure, which are composed via suitable tensor products. In the last Section 5, we revisit the Morita classification of star products for strong and covariant Morita equivalences, recalling Nikolai's work on the latter.

2 Ring-theoretic classifications of star products

2.1 Equivalences of star products and characteristic classes

We start by recalling the classical notion of equivalence for star products. We say that two star products $\star$ and $\star'$ on a Poisson manifold M are *equivalent* if there is a formal series $T = \mathsf{id} + \sum_{r=1}^{\infty} \hbar^r T_r$ of differential operators $T_r : C^\infty(M) \longrightarrow C^\infty(M)$ such that

$$(2.1) \qquad f \star' g = T^{-1}(Tf \star Tg) \quad \text{and} \quad T1 = 1,$$

for all $f, g \in C^\infty(M)[[\hbar]]$. We refer to T as an *equivalence transformation*. In particular, $\star$ and $\star'$ define isomorphic $\mathbb{C}[[\hbar]]$-algebra structures on $C^\infty(M)[[\hbar]]$.

Analogously, we call $\star$ and $\star'$ *diffeomorphic* if there is a Poisson diffeomorphism $\Phi \colon M \longrightarrow M$ with

$$(2.2) \qquad f \star' g = \Phi_*(\Phi^* f \star \Phi^* g),$$

for all $f, g \in C^\infty(M)[[\hbar]]$. Note that the fact that Φ preserves the Poisson structure is necessary if $\star$ and $\star'$ quantize the same Poisson bracket in first order. One may now verify that two star-product algebras $(C^\infty(M)[[\hbar]], \star)$ and $(C^\infty(M)[[\hbar]], \star')$ are isomorphic as algebras over $\mathbb{C}[[\hbar]]$ if and only if there is a Poisson diffeomorphism Φ and an equivalence transformation T such that, for all functions $f, g \in C^\infty(M)[[\hbar]]$, one has

$$(2.3) \qquad f \star' g = T^{-1}\Phi_*(\Phi^* Tf \star \Phi^* Tg).$$

The set of all star products on M is denoted by $\underline{\mathrm{Def}}(M)$, while $\underline{\mathrm{Def}}(M, \pi_1)$ denotes the set of star products for a fixed first-order Poisson bracket $\pi_1 \in \Gamma^\infty(\Lambda^2 TM)$. The equivalence transformations form a group under composition which acts on $\underline{\mathrm{Def}}(M)$ and leaves $\underline{\mathrm{Def}}(M, \pi_1)$ invariant. Hence we can form the orbit spaces for this group action, which we denote by $\mathrm{Def}(M)$ and $\mathrm{Def}(M, \pi_1)$, respectively. In other words, $\mathrm{Def}(M, \pi_1)$ is the set of classes of star products (up to equivalence) quantizing π_1.

For the classification of star products up to equivalence we rely on Kontsevich's formality theorem [25] and on the globalization of the formality map in [16]. In order to formulate the classification, recall that a *formal Poisson tensor* is a formal series $\pi = \hbar\pi_1 + \hbar^2\pi_2 + \cdots \in \hbar\Gamma^\infty(\Lambda^2 TM)[[\hbar]]$ with $[\![\pi, \pi]\!] = 0$, where we extend the Schouten bracket $[\![\cdot, \cdot]\!]$ $\hbar$-linearly. We denote the set of formal Poisson tensors on M by $\underline{\mathrm{FPoisson}}(M)$, and the subset of formal Poisson tensors with fixed first-order term π_1 by $\underline{\mathrm{FPoisson}}(M, \pi_1)$.

A formal vector field is a formal series $X = \hbar X_1 + \hbar^2 X_2 + \cdots \hbar\Gamma^\infty(TM)[[\hbar]]$. Since by definition a formal vector field starts in order $\hbar$, we can exponentiate its Lie derivative to get a well-defined operator

$$(2.4) \qquad \exp(\mathscr{L}_X) : \Gamma^\infty(\Lambda^\bullet TM)[[\hbar]] \longrightarrow \Gamma^\infty(\Lambda^\bullet TM)[[\hbar]],$$

preserving tensor degrees. Analogously, we can act on formal series of other kinds of tensor fields on M. By the Baker-Campbell-Hausdorff series one sees that the composition of $\exp(\mathscr{L}_X)$ and $\exp(\mathscr{L}_Y)$, for two formal vector fields X and Y, is again of the form $\exp(\mathscr{L}_Z)$ for a formal vector field $Z = \mathrm{BCH}(X, Y)$. Noticing that $\exp(\mathscr{L}_{-X})$ is the inverse of $\exp(\mathscr{L}_X)$, we see that the operators (2.4) form a group, called the *formal diffeomorphism group* of M and denoted by $\mathrm{FDiffeo}(M)$. If π is a formal Poisson tensor, then $\pi' = \exp(\mathscr{L}_X)(\pi)$ is still a formal Poisson tensor with the same first order term: $\pi'_1 = \pi_1$. Thus we get an action of $\mathrm{FDiffeo}(M)$ on the set of formal Poisson tensors which leaves $\underline{\mathrm{FPoisson}}(M, \pi_1)$ invariant. The orbit spaces of this group action are the equivalence classes of formal Poisson tensors up to formal diffeomorphisms, denoted by $\mathrm{FPoisson}(M)$ and $\mathrm{FPoisson}(M, \pi_1)$.

Kontsevich's formality theorem gives (among many other things) a construction of a star product $\star_\pi$ out of a given formal Poisson tensor π, once a global formality on M is chosen. The map $\pi \mapsto \star_\pi$ is such that, first, $\star_\pi$ quantizes π_1 as desired and, second, it induces a *bijection*

$$(2.5) \qquad \mathrm{FPoisson}(M, \pi_1) \ni [\pi] \; \mapsto \; [\star_\pi] \in \mathrm{Def}(M, \pi_1)$$

between the formal Poisson tensors deforming π_1, up to formal diffeomorphisms, and the formal star products quantizing π_1, up to equivalence. In other words, classes of star products in $\mathrm{Def}(M, \pi_1)$ are classified by elements in $\mathrm{FPoisson}(M, \pi_1)$. Also, using e.g. the globalized formality from [16], one can show that, for a Poisson diffeomorphism Φ, the star product $\Phi^*(\star_\pi)$ obtained from $\star_\pi$ as in (2.2) is equivalent to $\star_{\Phi^*\pi}$, though generally not equal; so (2.5) has a natural equivariance property relative to Poisson diffeomorphisms.

In the symplectic setting the above classification (2.5) can be made more concrete. In fact, the classification of star products on symplectic manifolds (M, ω) is prior to Kontsevich's work and can be phrased as follows: via the Fedosov construction [17] of symplectic star products one can associate to every formal series of closed two-forms $\Omega = \hbar\Omega_1 + \hbar^2\Omega_2 + \cdots \in \hbar\Gamma^\infty(\Lambda^2 T^*M)$ a star product $\star_\Omega$ such that any two $\star_\Omega$ and $\star_{\Omega'}$ are equivalent if and only if Ω and Ω' are cohomologous. Moreover, an inductive construction shows that for every star product $\star$ on (M, ω) there is an Ω such that $\star$ is equivalent to the Fedosov star product $\star_\Omega$. This leads to the classification of symplectic star products by their *Fedosov classes*,

$$(2.6) \qquad \mathrm{Def}(M, \omega) \ni [\star] \; \mapsto \; F(\star) = [\Omega] \in \hbar\mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})[[\hbar]],$$

where Ω is a formal series of closed two-forms such that $[\star] = [\star_\Omega]$. This point of view was developed by various authors, see [6, 30, 42].

Alternatively, one has an intrinsic classification not relying on the Fedosov construction but rather on a Cech cohomological argument: there is an intrinsic *characteristic class*

$$(2.7) \qquad c(\star) \in \frac{[\omega]}{\mathrm{i}\hbar} + \check{\mathrm{H}}^2(M, \mathbb{C})[[\hbar]]$$

such that $\star$ and $\star'$ are equivalent if and only if $c(\star) = c(\star')$, and any formal series in the affine space $\frac{[\omega]}{\mathrm{i}\hbar} + \check{\mathrm{H}}^2(M, \mathbb{C})[[\hbar]]$ arises as a characteristic class. Here the choice of $\frac{[\omega]}{\mathrm{i}\hbar}$ as the origin for the affine space is conventional. Remarkably, the construction of $c(\star)$ does not rely on any particular construction of star products but only on elementary facts about the Weyl star product on $\mathbb{R}^{2n}$ and a Cech cohomological patching on Darboux charts of (M, ω), see [15, 20] for this approach.

It is now a theorem of Nikolai that the two classes coincide after a trivial rescaling [33]: with the identification $\check{\mathrm{H}}^2(M, \mathbb{C}) = \mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})$, one gets

$$(2.8) \qquad\qquad c(\star) = \frac{[\omega] + F(\star)}{\mathrm{i}\hbar}.$$

Since symplectic manifolds are particular cases of Poisson manifolds, the classification of star products via Kontsevich's formality (2.5) should also match the classification via (2.7). This was verified in [14], where it was shown that Kontsevich's class $[\pi]$ of $\star$ is just the "inverse" of $c(\star)$. This makes sense as any representative of the formal series $c(\star)$ agrees, in lowest order, with the symplectic two-form ω; the fact that ω can be inverted to a Poisson tensor $\pi_1 = \omega^{-1}$ guarantees that the formal series can be inverted to a formal Poisson tensor.

2.2 Ring-theoretic Morita classification

We now consider a different classification problem in formal deformation quantization: viewing star products as unital $\mathbb{C}[[\hbar]]$-algebras, we discuss their classification up to (ring-theoretic) Morita equivalence. In subsequent sections we will present different ways in which Morita equivalence can be enhanced, and then revisit the classification of star products accordingly.

Let us briefly recall the notion of Morita equivalence [29] in its original form (see e.g. [26] for a textbook). Two unital algebras (over a fixed commutative, unital ground ring) $\mathcal{A}$ and $\mathcal{B}$ are *Morita equivalent* if there exists a $(\mathcal{B}, \mathcal{A})$-bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ which is "invertible" in the following sense: there is an $(\mathcal{A}, \mathcal{B})$-bimodule ${}_{\mathcal{A}}\mathcal{F}_{\mathcal{B}}$ for which there are bimodule isomorphisms

$$ {}_{\mathcal{A}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \cong {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}, \qquad {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{F}_{\mathcal{B}} \cong {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}. $$

Such a bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is referred to as an *equivalence bimodule*. As we will revisit Morita equivalence later in the paper, in more detail and from a broader perspective, we now only mention a few of its basic properties. First, as an equivalence relation among unital algebras, Morita equivalence is a nontrivial extension of the usual notion of algebra isomorphism: indeed, an isomorphism $\Phi \colon \mathcal{B} \longrightarrow \mathcal{A}$ gives rise to an equivalence bimodule which is simply $\mathcal{A}$ as a right $\mathcal{A}$-module, and where $\mathcal{B}$ acts on the left via Φ. Also, denoting by $\mathsf{Mod}(\mathcal{A})$ the category of left $\mathcal{A}$-modules, any equivalence bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ induces an equivalence of categories $\mathsf{Mod}(\mathcal{A}) \longrightarrow \mathsf{Mod}(\mathcal{B})$ via the tensor product $\otimes_{\mathcal{A}}$, and this is the sense in

which $\mathcal{A}$ and $\mathcal{B}$ have equivalent "representation theories". We finally remark that Morita's theorem provides a complete characterization of equivalence bimodules; in particular, it shows that they are finitely generated and projective over each algebra.

Example 2.1. We briefly discuss the equivalence bimodules ${}_\mathcal{B}\mathcal{E}_\mathcal{A}$ for which $\mathcal{A} = C^\infty(M)$ is the commutative algebra of complex-valued, smooth functions on a manifold M. It follows from the smooth version of Serre-Swan's theorem that, since any such equivalence bimodule is finitely generated and projective as a right $\mathcal{A}$-module, it must be given by the sections of a vector bundle $E \longrightarrow M$, on which $C^\infty(M)$ acts by pointwise multiplication; the algebra acting on the left is then necessarily isomorphic to $\Gamma^\infty(\mathsf{End}(E))$. In fact, any nonzero vector bundle defines an equivalence bimodule in this way. An auto-equivalence bimodule of $C^\infty(M)$ must be given by a line bundle $L \longrightarrow M$, since this is the only case where $C^\infty(M) \cong \Gamma^\infty(\mathsf{End}(L))$.

Going back to star products, the classification problem amounts to determining the conditions on the characteristic classes, as in (2.5) and (2.7), such that the corresponding star-product algebras are Morita equivalent. The easier part of the classification accounts for isomorphic star products: according to (2.3), if we mod out the equivalence transformations, we are still left with an action of Poisson diffeomorphisms on characteristic classes of star products whose orbits identify isomorphic ones. The more interesting part of the Morita classification comes from nontrivial equivalence bimodules. One may check that an equivalence bimodule for $\star$ and $\star'$ has a classical limit which remains an equivalence bimodule for the undeformed products. As seen in Example 2.1, such bimodules must be given by sections of line bundles. Hence the problem of Morita classification reduces to the question of which line bundles $L \longrightarrow M$ can be deformed into equivalence bimodules for star products. It turns out that one can always deform the sections $\Gamma^\infty(L)[[\hbar]]$ into a right $\star$-module in a unique way, up to equivalence [11]. This relies on the fact that the classical module is projective. Moreover, the endomorphisms $C^\infty(M) \cong \Gamma^\infty(\mathsf{End}(L))$ inherit a deformation $\star'$ from this procedure, in such a way that we get a deformed bimodule. The new star product $\star'$ quantizes the same Poisson bracket π_1 on M, see [11, 10]. The question is then how to compute the class of $\star'$ in terms of the class of $\star$ and the line bundle L.

We first mention the Morita classification for symplectic star products [12]:

Theorem 2.2 (Morita classification, symplectic case)**.** *Two star products $\star$ and $\star'$ on a symplectic manifold (M, ω) are Morita equivalent if and only if there is a symplectomorphism Φ such that*

$$(2.9) \qquad\qquad \Phi^* c(\star') - c(\star) \in 2\pi\mathrm{i}\mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{Z}).$$

In this case, the line bundle L with Chern class $c_1(L) = \frac{1}{2\pi\mathrm{i}}(\Phi^* c(\star') - c(\star))$ can be deformed into an equivalence bimodule for $\star$ and $\star'$.

More specifically, one obtains an equivalence bimodule for star products through a deformed bimodule structure on $\Gamma^\infty(L)[[\hbar]]$, where $\star$ acts on the right and $\star'$ acts (via Φ) on the left. Note that here only the image of the Chern class of L in *de Rham cohomology* matters; in particular, since torsion elements in $\check{H}^2(M, \mathbb{Z})$ vanish in $H^2_{\mathrm{dR}}(M, \mathbb{Z})$, they only account for isomorphic star products.

The previous theorem already hints at how the classification for star products on general Poisson manifolds should be: one should "invert" the relation $\Phi^* c(\star') = c(\star) + 2\pi i c_1(L)$ via a geometric series to get the corresponding relation for the Kontsevich classes. This heuristic reasoning first appeared in [23], where Morita equivalence was studied in the context of non-commutative gauge field theories.

To make this heuristics precise we have to elaborate on how two-forms act on Poisson structures. Given a formal Poisson structure

$$\pi = \hbar\pi_1 + \cdots \in \hbar\Gamma^\infty(\Lambda^2 TM)[[\hbar]],$$

we can equivalently view it, as usual, as a $\mathbb{C}[[\hbar]]$-linear bundle map

$$(2.10) \qquad \pi^\sharp \colon \Gamma^\infty(T^*M)[[\hbar]] \longrightarrow \hbar\Gamma^\infty(TM)[[\hbar]]$$

via $\pi^\sharp(\alpha) = \pi(\alpha, \cdot)$, where $\alpha \in \Gamma^\infty(T^*M)[[\hbar]]$. Analogously, given a two-form $B \in \Gamma^\infty(\Lambda^2 T^*M)[[\hbar]]$ we have a bundle map in the opposite direction

$$(2.11) \qquad B^\sharp \colon \Gamma^\infty(TM)[[\hbar]] \longrightarrow \Gamma^\infty(T^*M)[[\hbar]],$$

via $B^\sharp(X) = B(X, \cdot)$, for $X \in \Gamma^\infty(TM)[[\hbar]]$. Since we require π to start in first order of $\hbar$, the composition $B^\sharp\pi^\sharp$ is a $\mathbb{C}[[\hbar]]$-linear endomorphism of $\Gamma^\infty(T^*M)[[\hbar]]$ raising the $\hbar$-degree at least by one. Hence $\mathsf{id} + B^\sharp\pi^\sharp$ is necessarily invertible via a geometric series, so we may consider the inverse

$$(2.12) \qquad \left(\mathsf{id} + B^\sharp\pi^\sharp\right)^{-1} \colon \Gamma^\infty(T^*M)[[\hbar]] \longrightarrow \Gamma^\infty(T^*M)[[\hbar]].$$

We have the following results:

Proposition 2.3. *Let $B \in \Gamma^\infty(\Lambda^2 T^*M)[[\hbar]]$ and $\pi \in \hbar\Gamma^\infty(\Lambda^2 TM)[[\hbar]]$.*

1. *There exists a unique $\mathfrak{a}(B, \pi) \in \hbar\Gamma^\infty(\Lambda^2 TM)[[\hbar]]$ with $\mathfrak{a}(B, \pi)^\sharp = \pi^\sharp \circ (\mathsf{id} + B^\sharp\pi^\sharp)^{-1}$.*

2. *If π is a formal Poisson structure and B is closed then $\mathfrak{a}(B, \pi)$ is also a formal Poisson structure.*

3. *$\mathfrak{a}$ defines an action of the abelian group of formal series of closed two-forms on the set of formal Poisson structures.*

In analogy to the case without $\hbar$-powers, we call the map $\pi \mapsto \mathfrak{a}(B,\pi)$ a *gauge transformation* of π by the two-form B, see [40]; note that, in the purely geometric situation (with no powers in $\hbar$), the invertibility of $\mathsf{id} + B^\sharp \pi^\sharp$ is not automatic, depending on the choices of B and π.

A key feature of the action $\mathfrak{a}$ is that exact two-forms $B = \mathrm{d}\,A$, with $A \in \Gamma^\infty(T^*M)[[\hbar]]$, yield equivalent formal Poisson structures. Thus we obtain a well-defined action of the second de Rham cohomology $\mathrm{H}^2_{\mathrm{dR}}(M,\mathbb{C})[[\hbar]]$ on the equivalence classes of formal Poisson structures which preserves the lowest order term π_1. We denote this action by

$$(2.13) \qquad \mathfrak{a} \colon \mathrm{H}^2_{\mathrm{dR}}(M,\mathbb{C})[[\hbar]] \times \mathrm{FPoisson}(M,\pi_1) \longrightarrow \mathrm{FPoisson}(M,\pi_1).$$

This is the action which determines the Morita classification of star product [14]:

Theorem 2.4 (Morita classification, Poisson case). *Let $\star$ and $\star'$ be two star products on a Poisson manifold (M,π_1) with classes $[\pi]$ and $[\pi']$, respectively. Then $\star$ and $\star'$ are Morita equivalent if and only if there is a Poisson diffeomorphism Φ and an integral two-form B, $[B] \in 2\pi\mathrm{i}\mathrm{H}^2_{\mathrm{dR}}(M,\mathbb{Z})$, such that*

$$(2.14) \qquad\qquad \Phi^*[\pi'] = [\mathfrak{a}(B,\pi)].$$

As in Theorem 5.3, the corresponding line bundle with Chern class $c_1(L) = \frac{1}{2\pi\mathrm{i}}[B]$ can be deformed into an equivalence bimodule for $\star'$ and $\star$.

The construction of equivalence bimodules for star products can be refined in more specific geometric situations. We will mention two examples related to Nikolai's work, namely the cases of Kähler manifolds and cotangent bundles:

- For a Kähler manifold M, Fedosov's construction gives (at least) three canonical star products on M: the Weyl-ordered star product $\star_{\mathrm{Weyl}}$, the Wick star product $\star_{\mathrm{Wick}}$, and the anti-Wick star product $\star_{\overline{\mathrm{Wick}}}$. It was known that these three star product are not equivalent in general, and their characteristic classes are given by

  (2.15)
  $$c(\star_{\mathrm{Weyl}}) = \frac{[\omega]}{\mathrm{i}\hbar}, \quad c(\star_{\mathrm{Wick}}) = \frac{[\omega]}{\mathrm{i}\hbar} - \mathrm{i}\pi c_1(L_{\mathrm{can}}), \quad \text{and} \quad c(\star_{\overline{\mathrm{Wick}}}) = \frac{[\omega]}{\mathrm{i}\hbar} + \mathrm{i}\pi c_1(L_{\mathrm{can}}),$$

  where L_{can} denotes the canonical line bundle of M, i.e. the line bundle of holomorphic volume forms, see [24] as well as Nikolai's PhD thesis [32]. Thus we see from Theorem 2.2 that $\star_{\mathrm{Wick}}$ and $\star_{\overline{\mathrm{Wick}}}$ are always Morita equivalent, and they are Morita equivalent to $\star_{\mathrm{Weyl}}$ if and only if the canonical line bundle has a square root [34]. The construction of the deformed bimodule structure of L_{can} can be obtained from a rather explicit Fedosov construction. Also in [34] it was shown that for a holomorphic line bundle one can achieve deformed bimodule structures with the *separation of variables property*.

- For a cotangent bundle $M = T^*Q$, a line bundle $L \longrightarrow T^*Q$ is isomorphic to the pull-back of a line bundle on Q. Hence the curvature two-form of L corresponds to a closed two-form B on Q which has the physical interpretation of a *magnetic field*. If B is not exact, and thus L is not the trivial line bundle, then B corresponds to a magnetic monopole. The integrality condition in Theorem 2.2 can then be understood as Dirac's quantization condition for a magnetic monopole, giving a new interpretation of this condition in terms of Morita theory [12]. This result relates to previous work of Nikolai on the representation theory of star products, see [8, 9, 7], as well as his Diploma thesis [31]. We will come back to these results in Section 5.1.

3 Modules with additional structures

3.1 Inner products

We now consider additional properties of star-product algebras, beyond their ring structure, and discuss how they lead to enhanced notions of modules and representations. As mentioned in the introduction, we may restrict ourselves to Hermitian star products, which renders star product algebras with the structure of $*$-algebras, with involution given by complex conjugation of functions. We will also consider the order structure on the ring $\mathbb{R}[[\hbar]]$, which leads to various notions of positivity for star-product algebras. It will be convenient to work, more generally, in the following algebraic set-up: we will consider $*$-algebras over a ring $\mathsf{C} = \mathsf{R}(\mathsf{i})$, with $\mathsf{i}^2 = -1$ and R being an ordered ring. This framework encompasses Hermitian star product algebras (with $\mathsf{C} = \mathbb{C}[[\hbar]]$) and also C^*-algebras (with $\mathsf{C} = \mathbb{C}$).

Guided by the notions of Hilbert modules and strong Morita equivalence for C^*-algebras, see e.g. [38, 39, 27, 37], one considers the following. Let $\mathcal{A}$ be a $*$-algebra over C, and let $\mathcal{E}_\mathcal{A}$ be a right $\mathcal{A}$-module. We henceforth assume that all modules carry a compatible C-module structure such that all other structure maps are (multi-)linear over C. Even though this is not strictly necessary, we assume for simplicity that all algebras are unital and all modules are unital as well, i.e. the algebra unit acts as the identity on the module.

An $\mathcal{A}$-*valued inner product* is a map

$$(3.1) \qquad \langle \cdot, \cdot \rangle_\mathcal{A} : \mathcal{E}_\mathcal{A} \times \mathcal{E}_\mathcal{A} \longrightarrow \mathcal{A},$$

which is C-linear in the *second* argument, and such that $\langle x, y \cdot a \rangle_\mathcal{A} = \langle x, y \rangle_\mathcal{A} a$, for all $x, y \in \mathcal{E}_\mathcal{A}$ and $a \in \mathcal{A}$, and $\langle x, y \rangle_\mathcal{A} = (\langle y, x \rangle_\mathcal{A})^*$. We call $\langle \cdot, \cdot \rangle_\mathcal{A}$ *non-degenerate* if $\langle x, y \rangle_\mathcal{A} = 0$ for all $y \in \mathcal{E}_\mathcal{A}$ implies $x = 0$. Note that these inner products already make use of the $*$-involution.

In order to take into account the ordering of R, we proceed as follows. First, we call a linear functional $\omega \colon \mathcal{A} \longrightarrow \mathsf{C}$ *positive* if $\omega(a^*a) \geq 0$ for all $a \in \mathcal{A}$. In this case, ω satisfies a Cauchy-Schwarz inequality and behaves much like the

positive functionals in operator algebra theory. We use these positive functionals to define positivity of algebra elements: $a \in \mathcal{A}$ is *positive* if $\omega(a) \geq 0$ for all positive ω. In quantum physical terms this means that all the expectation values of the observable a are positive. Since this is all the information we can possibly get about the observable a in an operational way, this notion of "positivity by measurement" is well motivated by the desired applications in quantum physics. Standard arguments show that positive functionals form a convex cone in the dual of $\mathcal{A}$ which is stable under the operation $\omega \mapsto \omega_b$, with $\omega_b(a) = \omega(b^*ab)$ for every $b \in \mathcal{A}$. Moreover, the set of positive elements in $\mathcal{A}$, which we denote by $\mathcal{A}^+$, form a convex cone as well, stable under the maps $a \mapsto b^*ab$. Clearly, it contains the cone of "sums of squares" $\mathcal{A}^{++}$, i.e. those a which can be written as $a = \sum_{i=1}^{n} \alpha_i b_i^* b_i$ with $0 < \alpha_i \in \mathsf{R}$ and $b_i \in \mathcal{A}$. In general it is a nontrivial question to decide whether $\mathcal{A}^+ = \mathcal{A}^{++}$; for polynomials this is the famous Hilbert's 17th problem. For C^*-algebras one always has equality, a fact heavily relying on continuous spectral calculus.

We can now define the positivity requirements for an algebra-valued inner product. We call an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ *positive* if $\langle x, x \rangle_{\mathcal{A}} \in \mathcal{A}^+$, for all $x \in \mathcal{E}_{\mathcal{A}}$. To get better properties with respect to tensor products, it will be convenient to refine this notion and call $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ *completely positive* if, for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in \mathcal{E}_{\mathcal{A}}$, the matrix $(\langle x_i, x_j \rangle_{\mathcal{A}}) \in M_n(\mathcal{A})$ is positive. Here we use that $M_n(\mathcal{A})$ is naturally a $*$-algebra, so the notion of positivity makes sense. Such a right $\mathcal{A}$-module $\mathcal{E}_{\mathcal{A}}$ with completely positive and non-degenerate inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ will be called a (right) *pre-Hilbert $\mathcal{A}$-module*. If we only have a non-degenerate inner product, we call $\mathcal{E}_{\mathcal{A}}$ a (right) *inner-product $\mathcal{A}$-module*. It is clear that we can define an inner product on a left $\mathcal{A}$-module in an analogous way, replacing the $\mathcal{A}$-linearity in the second argument to the right by $\mathcal{A}$-linearity in the first argument to the left.

Let $\mathcal{B}$ be another $*$-algebra acting on $\mathcal{E}_{\mathcal{A}}$ from the left, such that we have a $(\mathcal{B}, \mathcal{A})$-bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. We always assume that the left $\mathcal{B}$-module structure is compatible with $\mathcal{E}_{\mathcal{A}}$, i.e., $\langle b \cdot x, y \rangle_{\mathcal{A}} = \langle x, b^* \cdot y \rangle_{\mathcal{A}}$ for all $b \in \mathcal{B}$ and $x, y \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. If the inner product is non-degenerate then we call this an *inner-product $(\mathcal{B}, \mathcal{A})$-bimodule*. If in addition $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ is completely positive, then we call ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ a *pre-Hilbert $(\mathcal{B}, \mathcal{A})$-bimodule*. Note that the two algebras $\mathcal{B}$ and $\mathcal{A}$ enter the picture in a non-symmetrical way.

Given two inner-product, or pre-Hilbert, bimodules ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and ${}_{\mathcal{B}}\mathcal{E}'_{\mathcal{A}}$, a *morphism* $T\colon {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \longrightarrow {}_{\mathcal{B}}\mathcal{E}'_{\mathcal{A}}$ is a bimodule morphism such that there exists a (necessarily unique) bimodule morphism $T^*\colon {}_{\mathcal{B}}\mathcal{E}'_{\mathcal{A}} \longrightarrow {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ with

$$(3.2) \qquad\qquad \langle x, Ty \rangle_{\mathcal{A}}{}' = \langle T^*x, y \rangle_{\mathcal{A}}$$

for all $x \in {}_{\mathcal{B}}\mathcal{E}'_{\mathcal{A}}$ and $y \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. We call T^* the adjoint of T. With these morphisms, one may consider the category of inner product $(\mathcal{B}, \mathcal{A})$-bimodules as well as the category of pre-Hilbert $(\mathcal{B}, \mathcal{A})$-bimodules. These categories define two possible

notions of "*-representation theory" for a *-algebra $\mathcal{B}$: we denote the categories of *-representations of $\mathcal{B}$ on inner-product $\mathcal{A}$-modules by *-$\mathsf{Mod}_{\mathcal{A}}(\mathcal{B})$, and on pre-Hilbert $\mathcal{A}$-modules by *-$\mathsf{Rep}_{\mathcal{A}}(\mathcal{B})$.

We conclude this section with some examples.

Example 3.1. For a unital *-algebra $\mathcal{A}$, consider the free right $\mathcal{A}$-module $\mathcal{A}^n$; we define the $\mathcal{A}$-valued inner product

$$(3.3) \qquad \langle x, y \rangle = \sum_{i=1}^{n} x_i^* y_i,$$

which is easily shown to be completely positive and non-degenerate. On $\mathcal{A}^n$ we have a natural left action of the matrix algebra $M_n(\mathcal{A})$, by matrix multiplication, which turns $\mathcal{A}^n$ into a pre-Hilbert $(M_n(\mathcal{A}), \mathcal{A})$-bimodule.

More generally, let $P = P^2 = P^* \in M_n(\mathcal{A})$ be a Hermitian idempotent matrix, i.e. a *projection*. Let us consider the projective right $\mathcal{A}$-module $P\mathcal{A}^n$, with the inner product given by the restriction of (3.3). Since P is a projection, we have $\langle Px, Py \rangle = \langle x, Py \rangle = \sum_{i=1}^{n} x_i^* P_{ij} y_j$, where $P_{ij} \in \mathcal{A}$ are the coefficients of P. One may check that this is a completely positive, non-degenerate inner product. If we consider $PM_n(\mathcal{A})P$ with its canonical *-algebra structure, then $P\mathcal{A}^n$ is a pre-Hilbert $(PM_n(\mathcal{A})P, \mathcal{A})$-bimodule. It is easy to see that $PM_n(\mathcal{A})P$ consists of *all* right $\mathcal{A}$-linear endomorphisms of $P\mathcal{A}^n$ in this case.

Example 3.2. Let us consider the geometric example $\mathcal{A} = C^\infty(M)$, as in Example 2.1. As mentioned there, a finitely-generated, projective module $P\mathcal{A}^n$ is, up to isomorphism, just the module of smooth sections $\Gamma^\infty(E)$ of a complex vector bundle $E \longrightarrow M$. A fiber metric h on E gives a non-degenerate inner product via

$$(3.4) \qquad \langle \psi, \phi \rangle (p) = h_p(\psi(p), \phi(p)),$$

for $p \in M$ and $\psi, \phi \in \Gamma^\infty(E)$. In this case we have not only non-degeneracy, but the map

$$(3.5) \qquad \Gamma^\infty(E) \ni \psi \mapsto \langle \psi, \cdot \rangle \in (\Gamma^\infty(E))^* = \Gamma^\infty(E^*)$$

from the right $\mathcal{A}$-module $\Gamma^\infty(E)$ into the dual left $\mathcal{A}$-module is bijective. In general, we call an inner product with this property *strongly non-degenerate*. Finally, note that writing $\Gamma^\infty(E) = PC^\infty(M)^n$ as a projective module amounts to establishing an isomorphism $E = \operatorname{im} P \subseteq M \times \mathbb{C}^n$ of E with a subbundle of the trivial bundle. Then $PM_n(C^\infty(M))P$ corresponds to the sections $\Gamma^\infty(\mathsf{End}(E))$ of the endomorphism bundle of E.

3.2 Hopf-algebra symmetries

We now discuss notions of (bi)modules when the algebras carry symmetries. In the C^*-algebraic framework this has been done for actions of locally compact groups

under the name of C^*-dynamical systems. We choose here a slightly more general notion of Hopf-algebra action so as to include infinitesimal actions of Lie algebras by derivations. Details can be found in [22].

Let H be a Hopf *-algebra over C, i.e. a Hopf algebra with a *-involution such that the coproduct Δ and the counit ϵ are *-homomorphisms, and such that $S(S(g)^*)^* = g$ for every $g \in H$, where S is the antipode of H. An H-*symmetry* of a *-algebra $\mathcal{A}$ is an action of H on $\mathcal{A}$, that we denote by

$$\rhd \colon H \times \mathcal{A} \longrightarrow \mathcal{A};$$

i.e. it is an H-module algebra structure, such that in addition we have $(g \rhd a)^* = S(g)^* \rhd a^*$. Suppose that all algebras in question have such a symmetry of a fixed Hopf *-algebra H. If we are given a (right) inner-product $\mathcal{A}$-module $\mathcal{E}_{\mathcal{A}}$, then we call it H-*covariant* (or H-equivariant) if we have an H-action on $\mathcal{E}_{\mathcal{A}}$ such that

$$(3.6) \qquad\qquad g \rhd (x \cdot a) = (g_{(1)} \rhd x) \cdot (g_{(2)} \rhd a)$$

and

$$(3.7) \qquad\qquad g \rhd \langle x, y \rangle_{\mathcal{A}} = \langle S(g_{(1)})^* \rhd x, g_{(2)} \rhd y \rangle_{\mathcal{A}},$$

where we use the Sweedler notation $\Delta(g) = g_{(1)} \otimes g_{(2)}$ for the coproduct. If we have an inner product $(\mathcal{B}, \mathcal{A})$-bimodule then we require an analogous compatibility for the left $\mathcal{B}$-module structure. Finally, morphisms between H-covariant bimodules are adjointable morphisms as above which, in addition, commute with the H-action. In this way we obtain the categories of H-covariant *-representations of a *-algebra $\mathcal{B}$ on H-covariant inner-product, or pre-Hilbert, $(\mathcal{B}, \mathcal{A})$-bimodules. We denote these categories by *-$\mathsf{Mod}_{\mathcal{A},H}(\mathcal{B})$ and *-$\mathsf{Rep}_{\mathcal{A},H}(\mathcal{B})$, respectively.

3.3 Tensor products

As we now see, all the notions of bimodule previously introduced can be seen as "generalized morphisms" between *-algebras; their composition is given by suitable tensor products, which we now discuss.

Let $_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}$ and $_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be inner-product, or pre-Hilbert, bimodules over the *-algebras $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, with or without H-symmetry. One defines an $\mathcal{A}$-valued inner product on the algebraic tensor product $_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ as follows: first, we set

$$(3.8) \qquad\qquad \langle \phi \otimes x, \psi \otimes y \rangle_{\mathcal{A}}^{\mathcal{F} \otimes \mathcal{E}} = \big\langle x, \langle \phi, \psi \rangle_{\mathcal{B}}^{\mathcal{F}} \cdot y \big\rangle_{\mathcal{A}}^{\mathcal{E}},$$

and we define an inner product by C-sesquilinear extension of this formula to all elements of the tensor product. Note that this is indeed well-defined on the tensor product over $\mathcal{B}$. It is not hard to check that $\langle \,\cdot\, , \,\cdot\, \rangle_{\mathcal{A}}^{\mathcal{F} \otimes \mathcal{E}}$ is an $\mathcal{A}$-valued inner product and the left $\mathcal{C}$-module structure is compatible with it. Slightly less trivial is the fact that this inner product is again completely positive, provided that

both inner products are completely positive, see [13, Thm. 4.7]. It may however be degenerate. To circumvent this problem, we mod out the tensor product by the subspace $({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})^{\perp}$ to get

$$(3.9) \qquad {}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \,\widehat{\otimes}_{\mathcal{B}} \, {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} := ({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}) \big/ ({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})^{\perp}.$$

It can be checked that this is an inner-product (resp. pre-Hilbert) $(\mathcal{C}, \mathcal{A})$-bimodule. Moreover, if all algebras and bimodules are H-covariant, then on the tensor product one defines an H-action in the usual way: $g \triangleright (\phi \otimes_{\mathcal{B}} x) = (g_{(1)} \triangleright \phi) \otimes_{\mathcal{B}} (g_{(2)} \triangleright x)$. This action passes to the quotient ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \,\widehat{\otimes}_{\mathcal{B}} \, {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and turns it into an H-covariant bimodule. All the above constructions are compatible with the morphisms we have specified, so we conclude that the tensor product defines functors

$$(3.10) \qquad \widehat{\otimes}_{\mathcal{B}} \colon \text{ }^*\text{-Mod}_{\mathcal{B},H}(\mathcal{C}) \times \text{ }^*\text{-Mod}_{\mathcal{A},H}(\mathcal{B}) \longrightarrow \text{ }^*\text{-Mod}_{\mathcal{A},H}(\mathcal{C})$$

as well as

$$(3.11) \qquad \widehat{\otimes}_{\mathcal{B}} \colon \text{ }^*\text{-Rep}_{\mathcal{B},H}(\mathcal{C}) \times \text{ }^*\text{-Rep}_{\mathcal{A},H}(\mathcal{B}) \longrightarrow \text{ }^*\text{-Rep}_{\mathcal{A},H}(\mathcal{C}),$$

where we can omit H for the versions without symmetry.

The tensor product $\widehat{\otimes}$ also enjoys the usual associativity properties, up to a canonical isomorphism. This means that we have an isomorphism

$$(3.12) \qquad \mathsf{asso} \colon \left({}_{\mathcal{D}}\mathcal{G}_{\mathcal{C}} \,\widehat{\otimes}_{\mathcal{C}} \, {}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}\right) \widehat{\otimes}_{\mathcal{B}} \, {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \longrightarrow {}_{\mathcal{D}}\mathcal{G}_{\mathcal{C}} \,\widehat{\otimes}_{\mathcal{C}} \left({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \,\widehat{\otimes}_{\mathcal{B}} \, {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}\right),$$

which respects all the structures on the bimodules, i.e. the inner products and, in the covariant case, the H-symmetry. Indeed, the usual associativity of the algebraic tensor product $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$ holds also on the quotients needed for $\widehat{\otimes}$, and respects all extra structures.

Since we have unital *-algebras, there is a canonical pre-Hilbert $(\mathcal{A}, \mathcal{A})$-bimodule given by ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$, with the inner product $\langle a, a' \rangle = a^* a'$. Note that the unit is needed to show that $\langle \cdot, \cdot \rangle$ is non-degenerate. This inner product is also *full*, in the sense that the span of all $\langle a, a' \rangle$ is the whole algebra $\mathcal{A}$. (More generally, we could use *-algebras which are idempotent and non-degenerate in the sense that $ab = 0$ for all b implies $a = 0$; then ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ would have the same properties.) If $\mathcal{A}$ is equipped with an H-symmetry, then ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ inherits this symmetry. These particular bimodules serve as "units" for the tensor product; i.e., there is a canonical isomorphism

$$(3.13) \qquad \mathsf{left} \colon {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}} \,\widehat{\otimes}_{\mathcal{B}} \, {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \longrightarrow {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$$

for every ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$, and similarly we have a canonical isomorphism

$$(3.14) \qquad \mathsf{right} \colon {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \,\widehat{\otimes}_{\mathcal{A}} \, {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \longrightarrow {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}},$$

respecting all the additional structures we have. Indeed, on the level of algebraic tensor products these maps are the usual ones, i.e. $b \otimes x \mapsto b \cdot x$. (For the

case of non-unital algebras, we have to add the conditions $\mathcal{B} \cdot {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} = {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \cdot \mathcal{A} = {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ for all the bimodules, so as to restore surjectivity of left and right.)

We observe that the isomorphisms asso, left, and right satisfy the usual coherence conditions, as the ones for the algebraic tensor product. This allows the construction of the following *bicategories* (weak 2-categories), see [4]. As objects we take unital *-algebras (more generally, we could work with non-degenerate and idempotent *-algebras). We can also add an H-symmetry for the *-algebras. For the 1-morphisms from $\mathcal{A}$ to $\mathcal{B}$, we take the inner-product (resp. pre-Hilbert) $(\mathcal{B}, \mathcal{A})$-bimodules, with H-symmetry if the *-algebras carry H-symmetry. For the 2-morphisms from ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ to ${}_{\mathcal{B}}\mathcal{E}'_{\mathcal{A}}$, we take the adjointable bimodule morphisms, which should be H-covariant in the presence of H-symmetries. The tensor product $\widehat{\otimes}$ together with the canonical maps asso, left, and right define a bicategory. We wind up with four possible flavors of bicategories of bimodules denoted by

1. $\underline{\mathrm{Bimod}}^*$ for inner-product bimodules,

2. $\underline{\mathrm{Bimod}}^{\mathrm{str}}$ for pre-Hilbert bimodules,

3. $\underline{\mathrm{Bimod}}^*_H$ for inner-product bimodules with H-symmetry,

4. $\underline{\mathrm{Bimod}}^{\mathrm{str}}_H$ for pre-Hilbert bimodules with H-symmetry.

For completeness, we mention that there are the ring-theoretic versions $\underline{\mathrm{Bimod}}$ and $\underline{\mathrm{Bimod}}_H$, where we only have algebras over C as objects but no *-involutions. In this case the tensor product is just the algebraic tensor product.

Important for us is the fact that in any bicategory we have a bigroupoid of *invertible* 1-*morphisms*. Here invertible means invertible with respect to the tensor product, up to 2-isomorphisms. This bigroupoid is called the *Picard bigroupoid* of the bicategory. In our situation, we have again four flavours of Picard groupoids:

1. The *-Picard bigroupoid $\underline{\mathrm{Pic}}^*$ is the bigroupoid of invertible 1-morphisms of $\underline{\mathrm{Bimod}}^*$.

2. The strong Picard bigroupoid $\underline{\mathrm{Pic}}^{\mathrm{str}}$ is the bigroupoid of invertible 1-morphisms in $\underline{\mathrm{Bimod}}^{\mathrm{str}}$.

3. The H-covariant *-Picard bigroupoid $\underline{\mathrm{Pic}}^*_H$ is the bigroupoid of invertible 1-morphisms of $\underline{\mathrm{Bimod}}^*_H$.

4. The H-covariant strong Picard bigroupoid $\underline{\mathrm{Pic}}^{\mathrm{str}}_H$ is the bigroupoid of invertible 1-morphisms in $\underline{\mathrm{Bimod}}^{\mathrm{str}}_H$.

Again, there are ring-theoretic versions of the Picard bigroupoid which we denote by $\underline{\mathrm{Pic}}$ and $\underline{\mathrm{Pic}}_H$, in the H-covariant case.

4 Strong and covariant Morita equivalences

Any bigroupoid corresponds to a groupoid, obtained through the identification of isomorphic 1-morphisms. For the Picard bigroupoids that we just introduced, we have to use *isometric* isomorphisms in order to respect all relevant structures. This leads to the following *Picard groupoids*: the $*$-Picard groupoid Pic^*, the strong Picard groupoid $\mathsf{Pic}^{\mathrm{str}}$, the H-covariant $*$-Picard groupoid Pic^*_H, and the H-covariant strong Picard groupoid $\mathsf{Pic}^{\mathrm{str}}_H$. These groupoids consist of the $*$-algebras as units and the equivalence classes of invertible bimodules (of the corresponding type) as arrows. In particular, for every $*$-algebra $\mathcal{A}$ we have the isotropy group of arrows starting and ending at $\mathcal{A}$. This is the *Picard group* of $\mathcal{A}$, which we denote by $\mathsf{Pic}^*(\mathcal{A})$, $\mathsf{Pic}^{\mathrm{str}}(\mathcal{A})$, $\mathsf{Pic}^*_H(\mathcal{A})$, and $\mathsf{Pic}^{\mathrm{str}}_H(\mathcal{A})$, depending on the case.
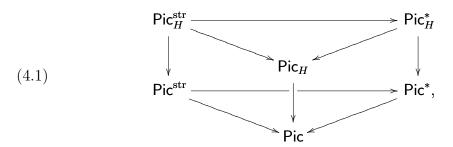
We now define the associated versions of Morita equivalence:

Definition 4.1 (Morita equivalence)**.** Two $*$-algebras over C are called

1. $*$-*Morita equivalent* if they are isomorphic in $\underline{\mathsf{Bimod}}^*$,

2. *strongly Morita equivalent* if they are isomorphic in $\underline{\mathsf{Bimod}}^{\mathrm{str}}$,

3. H-*covariantly $*$-Morita equivalent* if they are isomorphic in $\underline{\mathsf{Bimod}}^*_H$,

4. H-*covariantly strongly Morita* equivalent if they are isomorphic in $\underline{\mathsf{Bimod}}^{\mathrm{str}}_H$.

As usual, isomorphism of objects in a bicategory means that there is an invertible 1-morphism between them. Equivalently, two $*$-algebras are Morita equivalent in one of the above senses if and only if they are in the same orbit of the corresponding Picard groupoid. We also note that we have the ring-theoretic versions based on the Picard groupoids Pic and Pic_H, the former leading to the notion of Morita equivalence discussed in Section 2.2. A bimodule which is invertible, and hence defines a Morita equivalence, is also referred to as an *equivalence bimodule*, and a key problem is to characterize them in each case.

Note that forgetting the additional structures on bimodules (e.g. the complete positivity of inner products, the H-covariance, the inner products) preserves their invertibility. This gives the following diagram

(4.1)
$$
\begin{array}{ccc}
\mathsf{Pic}^{\mathrm{str}}_H & \longrightarrow & \mathsf{Pic}^*_H \\
\downarrow & \searrow\ \mathsf{Pic}_H\ \swarrow & \downarrow \\
\mathsf{Pic}^{\mathrm{str}} & \longrightarrow & \mathsf{Pic}^*, \\
& \searrow\ \mathsf{Pic}\ \swarrow &
\end{array}
$$

of commuting groupoid morphisms. Hence a lot of questions in Morita theory can be answered by first understanding the Picard groupoids Pic and Pic_H in

the ring-theoretic setting and, afterwards, investigating the kernels and images of these groupoid morphisms.

An immediate consequence of Morita equivalence is the equivalence of appropriate categories of modules:

Theorem 4.2 (Equivalence of representation theories). *Let $_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be a *-Morita equivalence bimodule, and let $\mathcal{D}$ be a fixed *-algebra. Then the functor*

$$(4.2) \qquad \mathsf{R}_{\mathcal{E}} = {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \,\widehat{\otimes}_{\mathcal{A}} \colon {}^{*}\text{-}\mathsf{Mod}_{\mathcal{D}}(\mathcal{A}) \longrightarrow {}^{*}\text{-}\mathsf{Mod}_{\mathcal{D}}(\mathcal{B})$$

*is an equivalence of categories. Analogous statements hold for a strong Morita equivalence bimodule, an H-covariant *-Morita equivalence bimodule, or an H-covariant strong Morita equivalence bimodule.*

The idea is to show that there are natural transformations from $\mathsf{R}_{\mathcal{A}}$ to the identity functor (via left) and from $\mathsf{R}_{\mathcal{F}} \circ \mathsf{R}_{\mathcal{E}}$ to $\mathsf{R}_{\mathcal{F}\widehat{\otimes}_{\mathcal{B}}\mathcal{E}}$ (via asso). Having the bicategory properties of $\underline{\mathsf{Bimod}}^{*}$, this is immediate.

Remark 4.3 (Picard groupoid actions). We can view Theorem 4.2 as a consequence of an *action* of the Picard groupoid on the representation theories of the *-algebras under consideration. In a similar way, many other Morita invariants can be viewed as arising from suitable actions of the Picard groupoid. Basic examples include the Picard groups themselves, the centers, the (H-equivariant) K-theory, and the lattices of certain *-ideals carrying information about the H-symmetry. We refer to [22] for a further discussion.

We now discuss how an equivalence bimodule actually looks like. Note that if $\mathcal{E}_{\mathcal{A}}$ is an inner-product right $\mathcal{A}$-module then we have particular *rank one* operators $\Theta_{x,y} \colon \mathcal{E}_{\mathcal{A}} \longrightarrow \mathcal{E}_{\mathcal{A}}$ defined by

$$(4.3) \qquad \Theta_{x,y}(z) = x \cdot \langle y, z \rangle_{\mathcal{A}},$$

for $x, y, z \in \mathcal{E}_{\mathcal{A}}$. From the properties of $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ we see that $\Theta_{x,y}$ is right $\mathcal{A}$-linear. Moreover, $\Theta_{x,y}$ has an adjoint operator explicitly given by $\Theta_{y,x}$. We denote by

$$(4.4) \qquad \mathfrak{F}(\mathcal{E}_{\mathcal{A}}) = \mathsf{C}\text{-}\mathrm{span}\left\{\Theta_{x,y} \mid x, y \in \mathcal{E}_{\mathcal{A}}\right\}$$

the *finite rank operators* on $\mathcal{E}_{\mathcal{A}}$. They form a *-algebra such that $\mathcal{E}_{\mathcal{A}}$ becomes an inner product $(\mathfrak{F}(\mathcal{E}_{\mathcal{A}}), \mathcal{A})$-bimodule. Moreover, if $\mathcal{E}_{\mathcal{A}}$ is equipped with an H-symmetry, then we get an induced *-action of H on $\mathfrak{F}(\mathcal{E}_{\mathcal{A}})$.

Theorem 4.4 (Equivalence bimodules). *Two unital *-algebras $\mathcal{A}$ and $\mathcal{B}$ are *-Morita equivalent if and only if there exists an inner product $(\mathcal{B}, \mathcal{A})$-bimodule $_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ such that*

1. The inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ is full (and necessarily strongly non-degenerate).

2. $\mathcal{B}$ is isomorphic to $\mathfrak{F}(\mathcal{E}_{\mathcal{A}})$ via the action map.

In this case $_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is equipped with a full $\mathcal{B}$-valued inner product $\Theta_{\cdot,\cdot}$ and $\mathcal{B} \cong \mathfrak{F}(\mathcal{E}_{\mathcal{A}})$ coincides with all adjointable operators on $\mathcal{E}_{\mathcal{A}}$. Moreover, $_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is finitely generated and projective as a right $\mathcal{A}$-module and as a left $\mathcal{B}$-module. If $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent, then $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ and $\Theta_{\cdot,\cdot}$ are, in addition, completely positive. In the H-covariant case, the bimodule carries an H-action compatible with both inner products.

For all cases, with additional effort, one also has non-unital formulations for idempotent and non-degenerate algebras. The *-Morita equivalence version is due to Ara [1], the strong Morita equivalence version comes from [13, Thm. 6.1], while the H-covariant versions were treated in [22].

5 Back to Morita classification of star products

We now revisit the Morita classification of star products, see Theorems 2.2 and 2.4, in light of the refined notions of Morita equivalence discussed in Section 4.

5.1 Strong Morita equivalence

It is known that, for unital C^*-algebras, ring-theoretic and strong Morita equivalences coincide, see [3]. It turns out that the same holds for Hermitian star-product algebras. The fact underlying this result is that, on any ring-theoretic equivalence bimodule between Hermitian star products, one can find suitable algebra-valued inner products. At the classical level of undeformed algebras, this follows from (3.4) since on every vector bundle we have a positive definite Hermitian fiber metric. Then one should verify that such fiber metrics can be deformed into algebra-valued inner product for $\star$. This fact was shown in [11] and treated more systematically in [13, Sect. 7 and Sect. 8], where the general relations between the ring-theoretic and the strong Picard groupoid, Pic and $\mathsf{Pic}^{\mathrm{str}}$, are studied in detail. The conclusion from [13, Thm. 8.9] can be formulated in terms of the groupoid morphisms (4.1):

Theorem 5.1 (Strong Morita equivalence of Hermitian star products)**.**

(a) *Within the class of Hermitian star products, the canonical groupoid morphism $\mathsf{Pic}^{\mathrm{str}} \longrightarrow \mathsf{Pic}$ is injective, and $\mathsf{Pic}^{\mathrm{str}}$ has the same orbits as Pic. In particular, two Hermitian star products are strongly Morita equivalent if and only if they are Morita equivalent.*

(b) *If $\star$ and $\star'$ are Morita equivalent Hermitian star products, then $\mathsf{Pic}^{\mathrm{str}}(\star, \star') \longrightarrow \mathsf{Pic}(\star, \star')$ is surjective if and only if all derivations of $\star$ are quasi-inner.*

In part (b), we use the notation $\mathsf{Pic}^{\mathrm{str}}(\star, \star')$ for the space of arrows in $\mathsf{Pic}^{\mathrm{str}}$ from $\star$ to $\star'$ (similarly for Pic); we also call a derivation D of $\star$ *quasi-inner* if it is of the form $D = \frac{1}{\mathrm{i}\hbar}[H, \,\cdot\,]_\star$, for some $H \in C^\infty(M)[[\hbar]]$. Hence, the coincidence of the ring-theoretic and strong Picard groups boils down to whether there are derivations which are not quasi-inner. In the symplectic case, it is known that all derivations are quasi-inner if and only if $\mathrm{H}^1_{\mathrm{dR}}(M, \mathbb{C}) = \{0\}$. So, although strong and ring-theoretic Morita equivalences define the same equivalence relation for Hermitian star products, the corresponding Picard groups are generally distinct.

In light of part (a) of the theorem, one may directly use Theorems 2.2 and 2.4 for a description of strongly Morita equivalent Hermitian star products in terms of their characteristic classes. We mention, for completeness, that a result of Nikolai [33, Sec. 5] characterizes symplectic Hermitian star products in terms of the classes (2.7): they must satisfy $\overline{c(\star)} = -c(\star)$, a property that is stable under Morita equivalence (c.f. Theorems 2.2). A similar characterization, extending Nikolai's result, should also hold for the classes (2.5) in the Poisson case.

Remark 5.2. We note that $*$-Morita equivalence of Hermitian star products falls into the same classification since, on a connected component of M, the (strongly non-degenerate) inner products on the sections of a line bundle can either be completely positive or completely negative.

As discussed in [12] and mentioned at the end of Section 2.2, strong Morita equivalence turns out to be related to Nikolai's work on the representation theory of star products on cotangent bundles $M = T^*Q$ [8, 9, 7]. More specifically, the usual Schrödinger type representation on functions on the configuration space Q requires a star product $\star$ with trivial class (i.e. without magnetic monopoles, see Section 2.2). In the presence of a magnetic monopole described by an integral two-form B, one can deform the associated line bundle on the cotangent bundle T^*Q into a *strong* Morita equivalence bimodule between $\star$ and a new star product $\star_B$. We can then use this equivalence bimodule to relate (pre-Hilbert) modules over $\star$ and $\star_B$ (see Theorem 4.2). In particular, tensoring this equivalence bimodule with the Schrödinger representation of $\star$ on $C_0^\infty(Q)[[\hbar]]$ yields a representation of $\star_B$ on the space of sections $\Gamma_0^\infty(L)[[\hbar]]$ of the line bundle L over Q determined by B. On the other hand, the star product $\star_B$ had been previously considered in Nikolai's joint work [7], where a representation of $\star_B$ on the space $\Gamma_0^\infty(L)[[\hbar]]$ was constructed directly, locally out of $\star$ by applying a local version of "minimal coupling" using the local potentials $A \in \Gamma^\infty(T^*U)$ of $B\big|_U = \mathrm{d}\,A$. It was shown in [12] that, modulo canonical identifications, both constructions agree: the representation of $\star_B$ on $\Gamma_0^\infty(L)[[\hbar]]$ from [7] exactly corresponds to the Schrödinger representation of $\star$ under strong Morita equivalence.

Still in this direction, we mention the unfinished project by Nikolai to transfer the ideas of the representation theory of star products on cotangent bundles to star products on general Lie algebroids. Building on [35], the plan was to construct representations and equivalence bimodules as in the cotangent bundle

case, thereby establishing the relation to the pseudo-differential operator algebraic quantizations in [36]. Nikolai was unfortunately not able to finish this project, but Nikolai's student Alexander Held took initial steps in his Diploma thesis.

5.2 Covariant Morita equivalence

We finally address covariant Morita equivalence for star products on symplectic manifolds; this was the subject of one of Nikolai's last joint projects. The general case of star products on Poisson manifolds is yet to be worked out, but should follow along the same lines, relying on Theorem 2.4 and equivariant formality maps [16].

Let (M, ω) be a symplectic manifold acted upon by a Lie algebra $\mathfrak{g}$; we assume the action to be symplectic, though not necessarily Hamiltonian. We will also assume that the action preserves a connection (and hence also a torsion-free symplectic connection). This is in fact a mild requirement: if the $\mathfrak{g}$-action comes from a symplectic action of a Lie group G and if this G-action is proper, then we always have such an invariant connection. But even in the non-proper case there are interesting examples where such a connection exists.

A star product $\star$ is called $\mathfrak{g}$-*invariant* if the fundamental vector fields $\xi_M \in \Gamma^\infty(TM)$ of the $\mathfrak{g}$-action act as derivations of $\star$ for all $\xi \in \mathfrak{g}$. One has a classification of $\mathfrak{g}$-invariant star products, up to $\mathfrak{g}$-invariant equivalence transformations [5]: every such star product is $\mathfrak{g}$-invariantly equivalent to a Fedosov star product $\star_\Omega$, where the closed two-form $\Omega \in \hbar\Gamma^\infty(\Lambda^2 T^*M)^{\mathfrak{g}}[[\hbar]]$ is $\mathfrak{g}$-invariant, and two such star products $\star_\Omega, \star'_\Omega$ are $\mathfrak{g}$-invariantly equivalent if and only if the corresponding two-forms Ω and Ω' are cohomologous in the *invariant* de Rham cohomology. Thus one can define a $\mathfrak{g}$-*invariant characteristic class* by

$$(5.1) \qquad c^{\mathfrak{g}}(\star) = \frac{[\omega] + [\Omega]}{i\hbar} \in \frac{[\omega]}{i\hbar} + \mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})^{\mathfrak{g}}[[\hbar]],$$

where $\mathrm{H}^\bullet_{\mathrm{dR}}(M, \mathbb{C})^{\mathfrak{g}}$ denotes the $\mathfrak{g}$-invariant de Rham cohomology of M.

Forgetting the invariance gives us a canonical map

$$(5.2) \qquad \mathrm{H}^\bullet_{\mathrm{dR}}(M, \mathbb{C})^{\mathfrak{g}} \longrightarrow \mathrm{H}^\bullet_{\mathrm{dR}}(M, \mathbb{C}).$$

We also need to consider the $\mathfrak{g}$-equivariant de Rham cohomology. We use the Cartan model, see e.g. [19]. Here we only need its Lie algebra version: the complex is

$$(5.3) \qquad \Omega^\bullet_{\mathfrak{g}}(M, \mathbb{C}) = \bigoplus_{k=0}^{\infty} \bigoplus_{2i+j=k} \left(\mathrm{Pol}^i(\mathfrak{g}) \otimes \Gamma^\infty(\Lambda^j T^*M)\right)^{\mathfrak{g}},$$

with the differential $\mathrm{d}_{\mathfrak{g}}$ given by $(\mathrm{d}_{\mathfrak{g}}\alpha)(\xi) = \mathrm{d}\,\alpha(\xi) + \mathrm{i}_{\xi_M}\alpha(\xi)$ for $\xi \in \mathfrak{g}$. In particular, for the second equivariant de Rham cohomology we have a two-form

part and a function part linear in $\mathfrak{g}$. Projecting on the two-form part, we get an induced map in cohomology

$$(5.4) \qquad\qquad \mathrm{H}^2_{\mathfrak{g}}(M, \mathbb{C}) \longrightarrow \mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})^{\mathfrak{g}}.$$

Using these canonical maps we can refine Theorem 2.2 as follows [21]:

Theorem 5.3. *Let (M, ω) be a symplectic manifold carrying a symplectic Lie algebra action of $\mathfrak{g}$ which preserves a connection. Let $\star$ and $\star'$ be two $\mathfrak{g}$-invariant star products (resp. Hermitian star products) on (M, ω). Then $\star$ and $\star'$ are $\mathfrak{g}$-covariantly (resp. strongly $\mathfrak{g}$-covariantly) Morita equivalent if and only if there exists a $\mathfrak{g}$-invariant symplectomorphism Φ such that $\Phi^* c^{\mathfrak{g}}(\star') - c^{\mathfrak{g}}(\star)$ is in the image of the first map in*

$$(5.5) \qquad\qquad \mathrm{H}^2_{\mathfrak{g}}(M, \mathbb{C}) \longrightarrow \mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})^{\mathfrak{g}} \longrightarrow \mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C}),$$

and maps to a $2\pi\mathrm{i}$-integral de Rham cohomology class under the second map.

As previously mentioned, a similar classification should hold in the Poisson case, based on Theorem 2.4 and on equivariant formality maps, as in [16]; we observe that, just as Theorem 5.3, the construction of equivariant formalities make use of $\mathfrak{g}$-invariant connections.

References

[1] Ara, P.: *Morita equivalence for rings with involution.* Alg. Rep. Theo. **2** (1999), 227–247.

[2] Bayen, F., Flato, M., Frønsdal, C., Lichnerowicz, A., Sternheimer, D.: *Deformation Theory and Quantization.* Ann. Phys. **111** (1978), 61–151.

[3] Beer, W.: *On Morita equivalence of nuclear C^*-algebras.* J. Pure Appl. Algebra **26**.3 (1982), 249–267.

[4] Bénabou, J.: *Introduction to Bicategories.* In: *Reports of the Midwest Category Seminar*, 1–77. Springer-Verlag, 1967.

[5] Bertelson, M., Bieliavsky, P., Gutt, S.: *Parametrizing Equivalence Classes of Invariant Star Products.* Lett. Math. Phys. **46** (1998), 339–345.

[6] Bertelson, M., Cahen, M., Gutt, S.: *Equivalence of Star Products.* Class. Quant. Grav. **14** (1997), A93–A107.

[7] Bordemann, M., Neumaier, N., Pflaum, M. J., Waldmann, S.: *On representations of star product algebras over cotangent spaces on Hermitian line bundles.* J. Funct. Anal. **199** (2003), 1–47.

[8] Bordemann, M., Neumaier, N., Waldmann, S.: *Homogeneous Fedosov Star Products on Cotangent Bundles I: Weyl and Standard Ordering with Differential Operator Representation.* Commun. Math. Phys. **198** (1998), 363–396.

[9] Bordemann, M., Neumaier, N., Waldmann, S.: *Homogeneous Fedosov star products on cotangent bundles II: GNS representations, the WKB expansion, traces, and applications.* J. Geom. Phys. **29** (1999), 199–234.

[10] Bursztyn, H.: *Poisson Vector Bundles, Contravariant Connections and Deformations.* In: Maeda, Y., Watamura, S. (eds.): *Noncommutative Geometry and String Theory.* [28], 26–37. Proceedings of the International Workshop on Noncommutative Geometry and String Theory.

[11] Bursztyn, H., Waldmann, S.: *Deformation Quantization of Hermitian Vector Bundles.* Lett. Math. Phys. **53** (2000), 349–365.

[12] Bursztyn, H., Waldmann, S.: *The characteristic classes of Morita equivalent star products on symplectic manifolds.* Commun. Math. Phys. **228** (2002), 103–121.

[13] Bursztyn, H., Waldmann, S.: *Completely positive inner products and strong Morita equivalence.* Pacific J. Math. **222** (2005), 201–236.

[14] Bursztyn, H., Dolgushev, V., Waldmann, S.: *Morita equivalence and characteristic classes of star products.* Crelle's J. reine angew. Math. **662** (2012), 95–163.

[15] Deligne, P.: *Déformations de l'Algèbre des Fonctions d'une Variété Symplectique: Comparaison entre Fedosov et DeWilde, Lecomte.* Sel. Math. New Series **1**.4 (1995), 667–697.

[16] Dolgushev, V. A.: *Covariant and equivariant formality theorems.* Adv. Math. **191** (2005), 147–177.

[17] Fedosov, B. V.: *Deformation Quantization and Index Theory.* Akademie Verlag, Berlin, 1996.

[18] Gerstenhaber, M.: *On the Deformation of Rings and Algebras.* Ann. Math. **79** (1964), 59–103.

[19] Guillemin, V. W., Sternberg, S.: *Supersymmetry and Equivariant de Rham Theory.* Springer-Verlag, Berlin, Heidelberg, New York, 1999.

[20] Gutt, S., Rawnsley, J.: *Equivalence of star products on a symplectic manifold; an introduction to Deligne's Čech cohomology classes.* J. Geom. Phys. **29** (1999), 347–392.

[21] JANSEN, S., NEUMAIER, N., SCHAUMANN, G., WALDMANN, S.: *Classification of Invariant Star Products up to Equivariant Morita Equivalence on Symplectic Manifolds*. Preprint **arXiv:1004:0875** (April 2010), 28 pages.

[22] JANSEN, S., WALDMANN, S.: *The H-covariant strong Picard groupoid*. J. Pure Appl. Alg. **205** (2006), 542–598.

[23] JURČO, B., SCHUPP, P., WESS, J.: *Noncommutative Line Bundles and Morita Equivalence*. Lett. Math. Phys. **61** (2002), 171–186.

[24] KARABEGOV, A. V., SCHLICHENMAIER, M.: *Identification of Berezin-Toeplitz deformation quantization*. J. reine angew. Math. **540** (2001), 49–76.

[25] KONTSEVICH, M.: *Deformation Quantization of Poisson manifolds*. Lett. Math. Phys. **66** (2003), 157–216.

[26] LAM, T. Y.: *Lectures on Modules and Rings*, vol. 189 in *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, Heidelberg, New York, 1999.

[27] LANCE, E. C.: *Hilbert C*-modules. A Toolkit for Operator algebraists*, vol. 210 in *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1995.

[28] MAEDA, Y., WATAMURA, S. (EDS.): *Noncommutative Geometry and String Theory*, vol. 144 in *Prog. Theo. Phys. Suppl.* Yukawa Institute for Theoretical Physics, 2001. Proceedings of the International Workshop on Noncommutative Geometry and String Theory.

[29] MORITA, K.: *Duality for modules and its applications to the theory of rings with minimum condition*. Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 83–142.

[30] NEST, R., TSYGAN, B.: *Algebraic Index Theorem*. Commun. Math. Phys. **172** (1995), 223–262.

[31] NEUMAIER, N.: *Sternprodukte auf Kotangentenbündeln und Ordnungs-Vorschriften*. master thesis, Fakultät für Physik, Albert-Ludwigs-Universität, Freiburg, 1998. Available at `http://idefix.physik.uni-freiburg.de/~nine/`.

[32] NEUMAIER, N.: *Klassifikationsergebnisse in der Deformationsquantisierung*. PhD thesis, Fakultät für Physik, Albert-Ludwigs-Universität, Freiburg, 2001. Available at `http://idefix.physik.uni-freiburg.de/~nine/`.

[33] NEUMAIER, N.: *Local ν-Euler Derivations and Deligne's Characteristic Class of Fedosov Star Products and Star Products of Special Type*. Commun. Math. Phys. **230** (2002), 271–288.

[34] NEUMAIER, N., WALDMANN, S.: *Morita equivalence bimodules for Wick type star products.* J. Geom. Phys. **47** (2003), 177–196.

[35] NEUMAIER, N., WALDMANN, S.: *Deformation Quantization of Poisson Structures Associated to Lie Algebroids.* SIGMA **5** (2009), 074.

[36] NISTOR, V., WEINSTEIN, A., XU, P.: *Pseudodifferential operators on differential groupoids.* Pacific J. Math. **189**.1 (1999), 117–152.

[37] RAEBURN, I., WILLIAMS, D. P.: *Morita Equivalence and Continuous-Trace C*-Algebras,* vol. 60 in *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1998.

[38] RIEFFEL, M. A.: *Induced representations of C*-algebras.* Adv. Math. **13** (1974), 176–257.

[39] RIEFFEL, M. A.: *Morita equivalence for C*-algebras and W*-algebras.* J. Pure. Appl. Math. **5** (1974), 51–96.

[40] ŠEVERA, P., WEINSTEIN, A.: *Poisson Geometry with a 3-Form Background.* In: MAEDA, Y., WATAMURA, S. (EDS.): *Noncommutative Geometry and String Theory.* [28], 145–154. Proceedings of the International Workshop on Noncommutative Geometry and String Theory.

[41] WALDMANN, S.: *Poisson-Geometrie und Deformationsquantisierung. Eine Einführung.* Springer-Verlag, Heidelberg, Berlin, New York, 2007.

[42] WEINSTEIN, A., XU, P.: *Hochschild cohomology and characteristic classes for star-products.* In: KHOVANSKIJ, A., VARCHENKO, A., VASSILIEV, V. (EDS.): *Geometry of differential equations. Dedicated to V. I. Arnold on the occasion of his 60th birthday,* 177–194. American Mathematical Society, Providence, 1998.

Henrique Bursztyn
IMPA
Estrada Dona Castorina 110
Jardim Botanico
Rio de Janeiro, 22460-320
Brazil
email: `henrique@impa.br`

Stefan Waldmann
KU Leuven
Department of Mathematics
Celestijnenlaan 200B bus 2400
BE-3001 Heverlee
Belgium
email: `Stefan.Waldmann@wis.kuleuven.be`

144