

Twisting Poisson algebras, coPoisson algebras and Quantization

by Martin Bordemann, Olivier Elchinger
and Abdenacer Makhlouf

Abstract

The purpose of this paper is to study twistings of Poisson algebras or bialgebras, coPoisson algebras or bialgebras and star-products. We consider Hom-algebraic structures generalizing classical algebraic structures by twisting the identities by a linear self map. We summarize the results on Hom-Poisson algebras and introduce Hom-coPoisson algebras and bialgebras. We show that there exists a duality between Hom-Poisson bialgebras and Hom-coPoisson bialgebras. A relationship between enveloping Hom-algebras endowed with Hom-coPoisson structures and corresponding Hom-Lie bialgebra structures is studied. Moreover we set quantization problems and generalize the notion of star-product. In particular, we characterize the twists for the Moyal-Weyl product for polynomials of several variables.

Introduction

The study of nonassociative algebras was originally motivated by certain problems in physics and other branches of mathematics. The first motivation to study nonassociative Hom-algebras comes from quasi-deformations of Lie algebras of vector fields, in particular q -deformations of Witt and Virasoro algebras [1, 9, 11, 22, 21].

Hom-Lie algebras were first introduced by Hartwig, Larsson and Silvestrov in order to describe q -deformations of Witt and Virasoro algebras using σ -derivations (see [20]). The corresponding associative type objects, called Hom-associative algebras were introduced by Makhlouf and Silvestrov in [31]. The enveloping algebras of Hom-Lie algebras were studied by Yau in [39]. The dual notions, Hom-coalgebras, Hom-bialgebras, Hom-Hopf algebras and Hom-Lie coalgebras, were studied first in [30, 32]. The study was enhanced in [41, 42] and done with a categorical point of view in [8]. Further developments and results could be found in [2, 3, 17, 18, 40, 43]. The notion of Hom-Poisson algebras appeared first in [31]

and then studied in [45]. The main feature of Hom-algebra and Hom-coalgebra structures is that the classical identities are twisted by a linear self map.

In this paper, we review the results on Hom-Poisson algebras and introduce the notions of Hom-coPoisson algebra and Hom-coPoisson bialgebra. We show that there is a duality between the Hom-coPoisson bialgebras and Hom-Poisson bialgebras, generalizing to Hom-setting the result in [35]. Moreover we set the quantization problems and generalize the notion of star-product. In particular, we characterize the twists for the Moyal-Weyl product for polynomials of several variables. In Section 1, we summarize the definitions of the Hom-algebra and Hom-coalgebra structures twisting classical structures of associative algebras, Lie algebras and their duals. We extend to Hom-structures the classical relationship between the algebra and its finite dual and provide an easy tool to obtain a Hom-structure from a classical structure and provide some examples. In Section 2, we review the main results on Hom-Poisson algebras from [31, 45] and construct some new examples. In Section 3, we introduce and study Hom-coPoisson algebras, Hom-coPoisson bialgebras and Hom-coPoisson Hopf algebras. We provide some key constructions and show a link between enveloping algebras (viewed as a Hom-bialgebra) endowed with a coPoisson structure and Hom-Lie bialgebras. Moreover, we show that there is a duality between Hom-Poisson bialgebra (resp. Hom-Poisson Hopf algebra) and Hom-coPoisson bialgebra (resp. Hom-coPoisson Hopf algebra). In Section 4, we review the 1-parameter formal deformation of Hom-algebras (resp. Hom-coalgebras) and their relationship to Hom-Poisson algebra (resp. Hom-coPoisson algebras). Then we state the quantization problem and define star-product in Hom-setting. Finally we study twistings of Moyal-Weyl star-product.

1 Twisting usual structures: Hom-algebras and Hom-coalgebras

Throughout this paper \mathbb{K} denotes a field of characteristic 0 and A is a \mathbb{K} -module. In the sequel we denote by σ the linear map $\sigma : A^{\otimes 3} \rightarrow A^{\otimes 3}$, defined as $\sigma(x_1 \otimes x_2 \otimes x_3) = x_3 \otimes x_1 \otimes x_2$ and by τ_{ij} linear maps $\tau : A^{\otimes n} \rightarrow A^{\otimes n}$ where $\tau_{ij}(x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_j \otimes \cdots \otimes x_n) = (x_1 \otimes \cdots \otimes x_j \otimes \cdots \otimes x_i \otimes \cdots \otimes x_n)$.

We mean by a Hom-algebra a triple (A, μ, α) in which $\mu : A^{\otimes 2} \rightarrow A$ is a linear map and $\alpha : A \rightarrow A$ is a linear self map. The linear map $\mu^{op} : A^{\otimes 2} \rightarrow A$ denotes the opposite map, i.e. $\mu^{op} = \mu \circ \tau_{12}$. A Hom-coalgebra is a triple (A, Δ, α) in which $\Delta : A \rightarrow A^{\otimes 2}$ is a linear map and $\alpha : A \rightarrow A$ is a linear self map. The linear map $\Delta^{op} : A \rightarrow A^{\otimes 2}$ denotes the opposite map, i.e. $\Delta^{op} = \tau_{12} \circ \Delta$. For a linear self-map $\alpha : A \rightarrow A$, we denote by α^n the n -fold composition of n copies of α , with $\alpha^0 \cong Id$. A Hom-algebra (A, μ, α) (resp. a Hom-coalgebra (A, Δ, α))

is said to be *multiplicative* if $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$ (resp. $\alpha^{\otimes 2} \circ \Delta = \Delta \circ \alpha$). The Hom-algebra is called *commutative* if $\mu = \mu^{op}$ and the Hom-coalgebra is called *cocommutative* if $\Delta = \Delta^{op}$.

Unless otherwise specified, we assume in this paper that the Hom-algebras and Hom-coalgebras are multiplicative. Classical algebras or coalgebras are also regarded as a Hom-algebras or Hom-coalgebras with identity twisting map. Given a Hom-algebra (A, μ, α) , we often use the abbreviation $\mu(x, y) = xy$ for $x, y \in A$. Likewise, for a Hom-coalgebra (A, Δ, α) , we will use Sweedler's notation $\Delta(x) = \sum_{(x)} x_1 \otimes x_2$ but often omit the symbol of summation. When the Hom-algebra (resp. Hom-coalgebra) is multiplicative, we also say that α is multiplicative for μ (resp. Δ).

In the following, we summarize the definitions and properties of usual Hom-structures, provide some examples and recall the twisting principle giving a way to turn a classical structure to Hom-structure using algebra morphisms.

1.1 Hom-associative algebras and Hom-Lie algebras

We recall the definitions of Hom-Lie algebras and Hom-associative algebras.

Definition 1.1. Let (A, μ, α) be a Hom-algebra.

1. The *Hom-associator* $\mathfrak{as}_{\mu, \alpha} : A^{\otimes 3} \rightarrow A$ is defined as

$$(1.1) \quad \mathfrak{as}_{\mu, \alpha} = \mu \circ (\mu \otimes \alpha - \alpha \otimes \mu).$$

2. The Hom-algebra A is called a *Hom-associative algebra* if it satisfies the Hom-associative identity

$$(1.2) \quad \mathfrak{as}_{\mu, \alpha} = 0.$$

3. A Hom-associative algebra A is called *unital* if there exists a linear map $\eta : \mathbb{K} \rightarrow A$ such that

$$(1.3) \quad \mu \circ (Id_A \otimes \eta) = \mu \circ (\eta \otimes Id_A) = \alpha.$$

The unit element is $1_A = \eta(1)$.

4. The *Hom-Jacobian* $J_{\mu, \alpha} : A^{\otimes 3} \rightarrow A$ is defined as

$$(1.4) \quad J_{\mu, \alpha} = \mu \circ (\alpha \otimes \mu) \circ (Id + \sigma + \sigma^2).$$

5. The Hom-algebra A is called a *Hom-Lie algebra* if it satisfies $\mu + \mu^{op} = 0$ and the Hom-Jacobi identity

$$(1.5) \quad J_{\mu, \alpha} = 0.$$

We recover the usual definitions of the associator, an associative algebra, the Jacobian, and a Lie algebra when the twisting map α is the identity map. In terms of elements $x, y, z \in A$, the Hom-associator and the Hom-Jacobian are

$$\begin{aligned} \mathbf{as}_{\mu,\alpha}(x, y, z) &= (xy)\alpha(z) - \alpha(x)(yz), \\ J_{\mu,\alpha}(x, y, z) &= \circlearrowleft_{x,y,z} [\alpha(x), [y, z]], \end{aligned}$$

where the bracket denotes the product and $\circlearrowleft_{x,y,z}$ denotes the cyclic sum on x, y, z .

Let (A, μ, α) and $A' = (A', \mu', \alpha')$ be two Hom-algebras (either Hom-associative or Hom-Lie). A linear map $f : A \rightarrow A'$ is a *morphism of Hom-algebras* if

$$\mu' \circ (f \otimes f) = f \circ \mu \quad \text{and} \quad f \circ \alpha = \alpha' \circ f.$$

It is said to be a *weak morphism* if holds only the first condition.

Remark 1.2. It was shown in [29] that the commutator of a Hom-associative algebra leads to a Hom-Lie algebra. The construction of the enveloping algebras of Hom-Lie algebras is described in [39].

Example 1.3. Let $\{x_1, x_2, x_3\}$ be a basis of a 3-dimensional vector space A over \mathbb{K} . The following multiplication μ and linear map α on $A = \mathbb{K}^3$ define a Hom-associative algebra over \mathbb{K} :

$$\begin{aligned} \mu(x_1, x_1) &= ax_1, & \mu(x_2, x_2) &= ax_2, \\ \mu(x_1, x_2) &= \mu(x_2, x_1) = ax_2, & \mu(x_2, x_3) &= bx_3, \\ \mu(x_1, x_3) &= \mu(x_3, x_1) = bx_3, & \mu(x_3, x_2) &= \mu(x_3, x_3) = 0, \end{aligned}$$

$$\alpha(x_1) = ax_1, \quad \alpha(x_2) = ax_2, \quad \alpha(x_3) = bx_3$$

where a, b are parameters in \mathbb{K} . This algebra is not associative when $a \neq b$ and $b \neq 0$, since

$$\mu(\mu(x_1, x_1), x_3) - \mu(x_1, \mu(x_1, x_3)) = (a - b)bx_3.$$

Example 1.4 (Jackson \mathfrak{sl}_2). The Jackson \mathfrak{sl}_2 is a q -deformation of the classical Lie algebra \mathfrak{sl}_2 . It carries a Hom-Lie algebra structure but not a Lie algebra structure by using Jackson derivations. It is defined with respect to a basis $\{x_1, x_2, x_3\}$ by the brackets and a linear map α such that

$$\begin{aligned} [x_1, x_2] &= -2qx_2, & \alpha(x_1) &= qx_1, \\ [x_1, x_3] &= 2x_3, & \alpha(x_2) &= q^2x_2, \\ [x_2, x_3] &= -\frac{1}{2}(1+q)x_1, & \alpha(x_3) &= qx_3, \end{aligned}$$

where q is a parameter in \mathbb{K} . If $q = 1$ we recover the classical \mathfrak{sl}_2 .

The following proposition gives an easy way to twist classical structures in Hom-structures.

Theorem 1.5 ([40]).

- (1) Let $A = (A, \mu)$ be an associative algebra and $\alpha : A \rightarrow A$ be a linear map which is multiplicative with respect to μ , i.e. $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$. Then $A_\alpha = (A, \mu_\alpha = \alpha \circ \mu, \alpha)$ is a Hom-associative algebra.
- (2) Let $A = (A, [,])$ be a Lie algebra and $\alpha : A \rightarrow A$ be a linear map which is multiplicative with respect to $[,]$, i.e. $\alpha \circ [,] = [,] \circ \alpha^{\otimes 2}$. Then $A_\alpha = (A, [,]_\alpha = \alpha \circ [,], \alpha)$ is a Hom-Lie algebra.

Proof. (1) We have

$$\begin{aligned} \mathfrak{as}_{\mu_\alpha, \alpha} &= \mu_\alpha \circ (\mu_\alpha \otimes \alpha - \alpha \otimes \mu_\alpha) \\ &= \alpha \circ \mu \circ \alpha^{\otimes 2} \circ (\mu \otimes Id - Id \otimes \mu) \\ &= \alpha^2 \circ \mathfrak{as}_{\mu, Id} = 0 \end{aligned}$$

since (A, μ) is associative, so A_α is a Hom-associative algebra.

- (2) We have $\forall x, y \in A$, $[y, x] = -[x, y]$ and

$$\begin{aligned} J_{[,]_\alpha, \alpha} &= [,]_\alpha \circ (\alpha \otimes [,]_\alpha) \circ (Id + \sigma + \sigma^2) \\ &= \alpha \circ [,] \circ \alpha^{\otimes 2} \circ (Id \otimes [,]) \circ (Id + \sigma + \sigma^2) \\ &= \alpha^2 \circ J_{[,], Id} = 0 \end{aligned}$$

since $[,]$ is a Lie bracket, so A_α is a Hom-Lie algebra. □

Remark 1.6. More generally, the categories of Hom-associative algebras and Hom-Lie algebras are closed under twisting self-weak morphisms. If $A = (A, \mu, \alpha)$ is a Hom-associative algebra (resp. Hom-Lie algebra) and β a weak morphism, then $A_\beta = (A, \mu_\beta = \beta \circ \mu, \beta \circ \alpha)$ is a Hom-associative algebra (resp. Hom-Lie algebra) as well (see [45]).

Example 1.7. To twist the usual Lie algebra \mathfrak{sl}_2 , generated by the elements e, f, h and relations $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$, we figure out morphisms $\alpha : \mathfrak{sl}_2 \rightarrow \mathfrak{sl}_2$ written as $\alpha = (\alpha_{ij})$ with respect to the basis (e, f, h) , by solving the equations $\alpha([x, y]) = [\alpha(x), \alpha(y)]$ with respect to the basis in the coefficients α_{ij} . We obtain $(\mathfrak{sl}_2)_\alpha$, Hom-Lie versions of \mathfrak{sl}_2 , with α given by

1. $\alpha(e) = \lambda e$, $\alpha(f) = \lambda^{-1} f$, $\alpha(h) = h$,
2. $\alpha(e) = \lambda f$, $\alpha(f) = \lambda^{-1} e$, $\alpha(h) = -h$,
3. $\alpha(e) = -\lambda^2 e + f + \lambda h$, $\alpha(f) = \frac{1}{2} e - \frac{1}{2\lambda^2} f + \frac{1}{2\lambda} h$, $\alpha(h) = \lambda e + \lambda^{-1} f$,

where λ is a nonzero element in \mathbb{K} .

1.2 Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras

In this section we summarize and describe some of basic properties of Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras which generalize the classical coalgebra, bialgebra and Hopf algebra structures.

Definition 1.8. A *Hom-coalgebra* is a triple (A, Δ, α) where A is a \mathbb{K} -module and $\Delta : A \rightarrow A \otimes A$, $\alpha : A \rightarrow A$ are linear maps.

A *Hom-coassociative coalgebra* is a Hom-coalgebra satisfying

$$(1.6) \quad (\alpha \otimes \Delta) \circ \Delta = (\Delta \otimes \alpha) \circ \Delta.$$

A Hom-coassociative coalgebra is said to be *counital* if there exists a map $\varepsilon : A \rightarrow \mathbb{K}$ satisfying

$$(1.7) \quad (id \otimes \varepsilon) \circ \Delta = \alpha \quad \text{and} \quad (\varepsilon \otimes id) \circ \Delta = \alpha.$$

Let (A, Δ, α) and (A', Δ', α') be two Hom-coalgebras (resp. Hom-coassociative coalgebras). A linear map $f : A \rightarrow A'$ is a *morphism of Hom-coalgebras* (resp. *Hom-coassociative coalgebras*) if

$$(f \otimes f) \circ \Delta = \Delta' \circ f \quad \text{and} \quad f \circ \alpha = \alpha' \circ f.$$

If furthermore the Hom-coassociative coalgebras admit counits ε and ε' , we have moreover $\varepsilon = \varepsilon' \circ f$.

The following theorem shows how to construct a new Hom-coassociative Hom-coalgebra starting with a Hom-coassociative Hom-coalgebra and a Hom-coalgebra morphism. We only need the coassociative comultiplication of the coalgebra.

Theorem 1.9. *Let $(A, \Delta, \alpha, \varepsilon)$ be a counital Hom-coalgebra and $\beta : A \rightarrow A$ be a weak Hom-coalgebra morphism. Then $(A, \Delta_\beta, \alpha \circ \beta, \varepsilon)$, where $\Delta_\beta = \Delta \circ \beta$, is a counital Hom-coassociative coalgebra.*

Proof. We show that $(A, \Delta_\beta, \alpha \circ \beta)$ satisfies the axiom (1.6).

Indeed, using the fact that $(\beta \otimes \beta) \circ \Delta = \Delta \circ \beta$, we have

$$\begin{aligned} (\alpha \circ \beta \otimes \Delta_\beta) \circ \Delta_\beta &= (\alpha \circ \beta \otimes \Delta \circ \beta) \circ \Delta \circ \beta \\ &= ((\alpha \otimes \Delta) \circ \Delta) \circ \beta^2 \\ &= ((\Delta \otimes \alpha) \circ \Delta) \circ \beta^2 \\ &= (\Delta_\beta \otimes \alpha \circ \beta) \circ \Delta_\beta. \end{aligned}$$

Moreover, the axiom (1.7) is also satisfied, since we have

$$(id \otimes \varepsilon) \circ \Delta_\beta = (id \otimes \varepsilon) \circ \Delta \circ \beta = \alpha \circ \beta = (\varepsilon \otimes id) \circ \Delta \circ \beta = (\varepsilon \otimes id) \circ \Delta_\beta.$$

□

Remark 1.10. The previous theorem shows that the category of coassociative Hom-coalgebras is closed under weak Hom-coalgebra morphisms. It leads to the following examples:

1. Let (A, Δ) be a coassociative coalgebra and $\beta : A \rightarrow A$ be a coalgebra morphism. Then (A, Δ_β, β) is a Hom-coassociative coalgebra.
2. Let (A, Δ, α) be a (multiplicative) coassociative Hom-coalgebra. For all non negative integer n , $(A, \Delta_{\alpha^n}, \alpha^{n+1})$ is a Hom-coassociative coalgebra.

In the following we show that there is a duality between Hom-associative and the Hom-coassociative structures (see [30, 32]).

Theorem 1.11. *Let (A, Δ, α) be a Hom-coassociative coalgebra. Then the dual $(A^*, \Delta^*, \alpha^*)$ is a Hom-associative algebra. Moreover, A^* is unital whenever A is counital.*

Proof. The product $\mu = \Delta^*$ is defined from $A^* \otimes A^*$ to A^* by

$$(fg)(x) = \Delta^*(f, g)(x) = \langle \Delta(x), f \otimes g \rangle = (f \otimes g)(\Delta(x)) = \sum_{(x)} f(x_1)g(x_2), \quad \forall x \in A$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between the vector space $A \otimes A$ and its dual vector space. For $f, g, h \in A^*$ and $x \in A$, we have

$$(fg)\alpha^*(h)(x) = \langle (\Delta \otimes \alpha) \circ \Delta(x), f \otimes g \otimes h \rangle$$

and

$$\alpha^*(f)(gh)(x) = \langle (\alpha \otimes \Delta) \circ \Delta(x), f \otimes g \otimes h \rangle.$$

So the Hom-associativity $\mu \circ (\mu \otimes \alpha^* - \alpha^* \otimes \mu) = 0$ follows from the Hom-coassociativity $(\Delta \otimes \alpha - \alpha \otimes \Delta) \circ \Delta = 0$.

Moreover, if A has a counit ε satisfying $(id \otimes \varepsilon) \circ \Delta = \alpha = (\varepsilon \otimes id) \circ \Delta$ then for $f \in A^*$ and $x \in A$ we have

$$(\varepsilon f)(x) = \sum_{(x)} \varepsilon(x_1)f(x_2) = \sum_{(x)} f(\varepsilon(x_1)x_2) = f(\alpha(x)) = \alpha^*(f)(x)$$

and

$$(f\varepsilon)(x) = \sum_{(x)} f(x_1)\varepsilon(x_2) = \sum_{(x)} f(x_1\varepsilon(x_2)) = f(\alpha(x)) = \alpha^*(f)(x)$$

which shows that ε is the unit in A^* . □

The dual of a Hom-algebra (A, μ, α) is not always a Hom-coalgebra, because the coproduct does not land in the good space: $\mu^* : A^* \rightarrow (A \otimes A)^* \supsetneq A^* \otimes A^*$. It is the case if the Hom-algebra is finite dimensional, since $(A \otimes A)^* = A^* \otimes A^*$.

In the general case, for any algebra A , define

$$A^\circ = \{f \in A^*, f(I) = 0 \text{ for some cofinite ideal } I \text{ of } A\},$$

where a *cofinite ideal* I is an ideal $I \subset A$ such that A/I is finite dimensional.

A° is a subspace of A^* since it is closed under multiplication by scalars and the sum of two elements of A° is again in A° since the intersection of two cofinite ideals is again a cofinite ideal. If A is finite dimensional, of course $A^\circ = A^*$.

For an (Hom-)algebra map $f : A \rightarrow B$ we note $f^\circ := f^*|_{B^\circ} : B^\circ \rightarrow A^\circ$. Since $A^\circ \otimes A^\circ = (A \otimes A)^\circ$ (see [38, Lemma 6.0.1]) the dual $\mu^* : A^* \rightarrow (A \otimes A)^*$ of the multiplication $\mu : A \otimes A$ satisfies $\mu^*(A^\circ) \subset A^\circ \otimes A^\circ$. Indeed, for $f \in A^*$, $x, y \in A$, we have $\langle \mu^*(f), x \otimes y \rangle = \langle f, xy \rangle$. So if I is a cofinite ideal such that $f(I) = 0$, then $I \otimes A + A \otimes I$ is a cofinite ideal of $A \otimes A$ which vanish on $\mu^*(f)$. Similarly we have $\alpha^\circ(A^\circ) \subset A^\circ$.

Define $\Delta = \mu^\circ = \mu^*|_{A^\circ}$, $\alpha^\circ = \alpha^*|_{A^\circ}$ and $\varepsilon : A^\circ \rightarrow \mathbb{K}$ by $\varepsilon(f) = f(1)$.

Theorem 1.12. *Let (A, μ, α) be a multiplicative Hom-associative algebra. Then $(A^\circ, \Delta, \alpha^\circ)$ is a Hom-coassociative coalgebra. Moreover, it is counital if A is unital.*

Proof. The coproduct Δ is defined from A° to $A^\circ \otimes A^\circ$ by

$$\Delta(f)(x \otimes y) = \mu^*|_{A^\circ}(f)(x \otimes y) = \langle \mu(x \otimes y), f \rangle = f(xy), \quad x, y \in A.$$

For $f, g, h \in A^\circ$ and $x, y \in A$, we have

$$(\Delta \circ \alpha^\circ) \circ \Delta(f)(x \otimes y \otimes z) = \langle \mu \circ (\mu \otimes \alpha)(x \otimes y \otimes z), f \rangle$$

and

$$(\alpha^\circ \circ \Delta) \circ \Delta(f)(x \otimes y \otimes z) = \langle \mu \circ (\alpha \otimes \mu)(x \otimes y \otimes z), f \rangle.$$

So the Hom-coassociativity $(\Delta \otimes \alpha^\circ - \alpha^\circ \otimes \Delta) \circ \Delta = 0$ follows from the Hom-associativity $\mu \circ (\mu \otimes \alpha - \alpha \otimes \mu) = 0$.

Moreover, if A has a unit η satisfying $\mu \circ (id \otimes \eta) = \alpha = \mu \circ (\eta \otimes id)$ then for $f \in A^\circ$ and $x \in A$ we have

$$(\varepsilon \otimes id) \circ \Delta(f)(x) = f(1 \cdot x) = f(\alpha(x)) = \alpha^\circ(f)(x)$$

and

$$(id \otimes \varepsilon) \circ \Delta(f)(x) = f(x \cdot 1) = f(\alpha(x)) = \alpha^\circ(f)(x)$$

which shows that $\varepsilon : A^\circ \rightarrow \mathbb{K}$, $f \mapsto f(1)$ is the counit in A° . \square

If the Hom-associative algebra is finite dimensional then for any Hom-associative algebra the dual is provided with a structure of Hom-coassociative coalgebra.

Definition 1.13. A *Hom-bialgebra* is a tuple $(A, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ where

1. (A, μ, α, η) is a Hom-associative algebra with a unit η .
2. $(A, \Delta, \beta, \varepsilon)$ is a Hom-coassociative coalgebra with a counit ε .
3. The linear maps Δ and ε are compatible with the multiplication μ and the unit η , that is

$$(1.8) \quad \Delta(e) = e \otimes e \quad \text{where } e = \eta(1),$$

$$(1.9) \quad \Delta(\mu(x \otimes y)) = \Delta(x) \bullet \Delta(y) = \sum_{(x)(y)} \mu(x_1 \otimes y_1) \otimes \mu(x_2 \otimes y_2),$$

$$(1.10) \quad \varepsilon(e) = 1,$$

$$(1.11) \quad \varepsilon(\mu(x \otimes y)) = \varepsilon(x) \varepsilon(y),$$

$$(1.12) \quad \varepsilon \circ \alpha(x) = \varepsilon(x),$$

where the bullet \bullet denotes the multiplication on tensor product.

If $\alpha = \beta$ the Hom-bialgebra is denoted $(A, \mu, \eta, \Delta, \varepsilon, \alpha)$.

Combining previous observations, with only one twisting map, we obtain:

Proposition 1.14. *Let (A, μ, Δ, α) be a multiplicative Hom-bialgebra. Then the finite dual $(A^\circ, \mu^\circ, \Delta^\circ, \alpha^\circ)$ is a Hom-bialgebra as well.*

Remark 1.15.

1. Given a Hom-bialgebra $(A, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$, it is shown in [30, 32] that the vector space $\text{Hom}(A, A)$ with the multiplication given by the convolution product carries a structure of Hom-associative algebra.
2. An endomorphism S of A is said to be an *antipode* if it is the inverse of the identity over A for the Hom-associative algebra $\text{Hom}(A, A)$ with the multiplication given by the convolution product defined by

$$f \star g = \mu \circ (f \otimes g) \Delta$$

and the unit being $\eta \circ \varepsilon$.

3. A *Hom-Hopf algebra* is a Hom-bialgebra with an antipode.

By combining Theorems 1.5 and 1.9, we obtain:

Proposition 1.16. *Let (A, μ, Δ, α) be a Hom-bialgebra and $\beta : A \rightarrow A$ be a Hom-bialgebra morphism commuting with α . Then $(A, \mu_\beta, \Delta_\beta, \beta \circ \alpha)$ is a Hom-bialgebra.*

Notice that the category of Hom-bialgebra is not closed under weak Hom-bialgebra morphisms.

Example 1.17 (Universal enveloping Hom-algebra). Given a multiplicative Hom-associative algebra $A = (A, \mu, \alpha)$, one can associate to it a multiplicative Hom-Lie algebra $HLie(A) = (A, [,], \alpha)$ with the same underlying module (A, α) and the bracket $[,] = \mu - \mu^{op}$. This construction gives a functor $HLie$ from multiplicative Hom-associative algebras to multiplicative Hom-Lie algebras. In [39], Yau constructed the left adjoint U_{HLie} of $HLie$. He also made some minor modifications in [42] to take into account the unital case.

The functor U_{HLie} is defined as

$$(1.13) \quad U_{HLie}(A) = F_{HNAs}(A)/I^\infty \quad \text{with} \quad F_{HNAs}(A) = \bigoplus_{n \geq 1} \bigoplus_{\tau \in T_n^{wt}} A_\tau^{\otimes n}$$

for a multiplicative Hom-Lie algebra $(A, [,], \alpha)$. Here T_n^{wt} is the set of weighted n -trees encoding the multiplication of elements (by trees) and twisting by α (by weights), $A_\tau^{\otimes n}$ is a copy of $A^{\otimes n}$ and I^∞ is a certain submodule of relations build in such a way that the quotient is Hom-associative.

Moreover, the comultiplication $\Delta : U_{HLie}(A) \rightarrow U_{HLie}(A) \otimes U_{HLie}(A)$ defined by $\Delta(x) = \alpha(x) \otimes 1 + 1 \otimes \alpha(x)$ equips the multiplicative Hom-associative algebra $U_{HLie}(A)$ with a structure of Hom-bialgebra.

1.3 Hom-Lie coalgebras and Hom-Lie bialgebras

As it is the case for the Hom-associative algebras, the Hom-Lie algebras also have a dualized version, Hom-Lie coalgebras. They share the same kind of properties. We review here the principal results, similar results can be found in [44].

Definition 1.18. A *Hom-Lie coalgebra* (A, Δ, α) is a (multiplicative) Hom-coalgebra satisfying $\Delta + \Delta^{op} = 0$ and the Hom-coJacobi identity

$$(1.14) \quad (Id + \sigma + \sigma^2) \circ (\alpha \otimes \Delta) \circ \Delta = 0.$$

We call Δ the cobracket.

We recover Lie coalgebra when $\alpha = Id$. Just like (co)associative (co)algebras we have the following dualization properties.

Theorem 1.19.

1. Let (A, Δ, α) be a Hom-Lie coalgebra. Then $(A^*, \Delta^*, \alpha^*)$ is a Hom-Lie algebra.
2. Let $(A, [,], \alpha)$ be a multiplicative Hom-Lie algebra. Then $(A^\circ, [,]^\circ, \alpha^\circ)$ is a Hom-Lie coalgebra.

Notice that if A is finite dimensional then we may remove the assumption of multiplicativity.

The twisting principle also works, showing that the category of Hom-Lie coalgebras is closed under weak Hom-coalgebras morphisms.

Theorem 1.20. *Let (A, Δ, α) be a Hom-Lie coalgebra and $\beta : A \rightarrow A$ a weak Hom-coalgebra morphism. Then $(A, \Delta_\beta = \Delta \circ \beta, \alpha \circ \beta)$ is a Hom-Lie coalgebra.*

Proof. We have $\Delta_\beta + \Delta_\beta^{op} = (\Delta + \Delta^{op}) \circ \beta = 0$, and

$$\begin{aligned} (Id + \sigma + \sigma^2) \circ (\alpha \circ \beta \otimes \Delta_\beta) \circ \Delta_\beta &= (Id + \sigma + \sigma^2) \circ (\alpha \circ \beta \otimes \Delta \circ \beta) \circ \Delta \circ \beta \\ &= (Id + \sigma + \sigma^2) \circ (\alpha \otimes \Delta) \circ \Delta \circ \beta^2 \\ &= 0. \end{aligned}$$

□

The previous theorem can be used to construct Hom-Lie coalgebras.

Corollary 1.21.

1. *Let (A, Δ) be a Lie coalgebra and $\beta : A \rightarrow A$ be a Lie coalgebra morphism. Then (A, Δ_β, β) is a Hom-Lie coalgebra.*
2. *Let (A, Δ, α) be a (multiplicative) Hom-Lie coalgebra. For all non negative integer n , $(A, \Delta_{\alpha^n}, \alpha^{n+1})$ is a Hom-Lie coalgebra.*

The Hom-Lie bialgebra structure was introduced first in [44]. The definition presented below is slightly more general. They border the class defined by Yau and permit to consider the compatibility condition for different A -valued cohomology of Hom-Lie algebras.

Definition 1.22. A Hom-Lie bialgebra is a tuple $(A, [,], \alpha, \Delta, \beta)$ where

1. $(A, [,], \alpha)$ is a Hom-Lie algebra.
2. (A, Δ, β) is a Hom-Lie coalgebra.
3. The following compatibility condition holds for $x, y \in A$:

$$(1.15) \quad \Delta([x, y]) = ad_{\alpha(x)}(\Delta(y)) - ad_{\alpha(y)}(\Delta(x)),$$

where the adjoint map $ad_x : A^{\otimes n} \rightarrow A^{\otimes n}$ ($n \geq 2$) is given by

$$(1.16) \quad ad_x(y_1 \otimes \cdots \otimes y_n) = \sum_{i=1}^n \beta(y_1) \otimes \cdots \otimes \beta(y_{i-1}) \otimes [x, y_i] \otimes \beta(y_{i+1}) \otimes \cdots \otimes \beta(y_n).$$

A morphism $f : A \rightarrow A'$ of Hom-Lie bialgebras is a linear map commuting with α and β such that $f \circ [,] = [,] \circ f^{\otimes 2}$ and $\Delta \circ f = f^{\otimes 2} \circ \Delta$.

If $\alpha = \beta = Id$ we recover Lie bialgebras and if $\alpha = \beta$, we recover the class defined in [44], these Hom-Lie bialgebras are denoted $(A, [,], \Delta, \alpha)$.

In terms of elements, the compatibility condition (1.15) writes

$$(1.17) \quad \begin{aligned} \Delta([x, y]) &= ad_{\alpha(x)}(\Delta(y)) - ad_{\alpha(y)}(\Delta(x)) \\ &= [\alpha(x), y_1] \otimes \beta(y_2) + \beta(y_1) \otimes [\alpha(x), y_2] \\ &\quad - [\alpha(y), x_1] \otimes \beta(x_2) - \beta(x_1) \otimes [\alpha(y), x_2]. \end{aligned}$$

Remark 1.23. If $\alpha = \beta$, the compatibility condition (1.17) is equivalent to say that Δ is a 1-cocycle with respect to α^0 -adjoint cohomology of Hom-Lie algebras and for $\beta = Id$, it corresponds to α^{-1} -adjoint cohomology, (see [37] and [2]).

The following Proposition generalizes slightly [44, Theorem 3.5].

Proposition 1.24. *Let $(A, [,], \Delta, \alpha)$ be a Hom-Lie bialgebra and $\beta : A \rightarrow A$ a Hom-Lie bialgebra morphism commuting with α . Then $(A, [,]_\beta = \beta \circ [,], \Delta_\beta = \Delta \circ \beta, \alpha \circ \beta)$ is a Hom-Lie bialgebra.*

Proof. We already know that $(A, [,]_\beta, \beta \circ \alpha)$ is a Hom-Lie algebra and that $(A, \Delta_\beta, \alpha \circ \beta)$ is a Hom-Lie coalgebra. It remains to prove the compatibility condition (1.17) for Δ_β and $[,]_\beta$, with the twisting map $\alpha \circ \beta = \beta \circ \alpha$. On one side, we have

$$\Delta_\beta([x, y]) = \Delta \circ \beta^2 \circ [x, y] = (\beta^{\otimes 2})^2 \circ \Delta([x, y]),$$

since $\Delta \circ \beta = \beta^{\otimes 2} \circ \Delta$. Using in addition $\beta \circ [,] = [,] \circ \beta^{\otimes 2}$ and the fact that α and β commute, we have

$$\begin{aligned} ad_{\alpha \circ \beta(x)}(\Delta_\beta(y)) &= ad_{\alpha \circ \beta(x)}(\beta(y_1) \otimes \beta(y_2)) \\ &= [\alpha \circ \beta(x), \beta(y_1)]_\beta \otimes \alpha \circ \beta^2(y_2) + \alpha \circ \beta^2(y_1) \otimes [\alpha \circ \beta(x), \beta(y_2)]_\beta \\ &= (\beta^{\otimes 2})^2 ([\alpha(x), y_1] \otimes \alpha(y_2) + \alpha(y_1) \otimes [\alpha(x), y_2]). \end{aligned}$$

It follows that $\Delta_\beta([x, y]) = ad_{\alpha \circ \beta(x)}(\Delta_\beta(y)) - ad_{\alpha \circ \beta(y)}(\Delta_\beta(x))$ as wished. \square

As for the Hom-bialgebra, the category of Hom-Lie bialgebra is not closed under weak Hom-Lie bialgebra morphisms.

Hom-Lie bialgebra can be dualized. We obtain the following proposition generalized the result stated in [44] for finite dimensional case and using natural pairing.

Proposition 1.25. *Let $(A, [,], \alpha, \Delta, \beta)$ be a multiplicative Hom-Lie bialgebra. Then the finite dual $(A^\circ, [,]^\circ, \alpha^\circ, \Delta^\circ, \beta^\circ)$ is a Hom-Lie bialgebra as well.*

2 Hom-Poisson algebras

The Hom-Poisson algebras were introduced in [31], where they emerged naturally in the study of 1-parameter formal deformations of commutative Hom-associative algebras. Then this structure was studied in [45]. It is shown that they are closed under twisting by suitable self maps and under tensor products. Moreover it is shown that (de)polarization Hom-Poisson algebras are equivalent to admissible Hom-Poisson algebras, each of which has only one binary operation. We survey in this section the fundamental results and provide examples.

Definition 2.1. A *Hom-Poisson algebra* is a tuple $(A, \mu, \{ , \}, \alpha)$ consisting of

- (1) a commutative Hom-associative algebra (A, μ, α) and
- (2) a Hom-Lie algebra $(A, \{ , \}, \alpha)$

such that the Hom-Leibniz identity

$$(2.1) \quad \{ , \} \circ (\mu \otimes \alpha) = \mu \circ (\alpha \otimes \{ , \}) + (\{ , \} \otimes \alpha) \circ \tau_{23}$$

is satisfied.

In a Hom-Poisson algebra $(A, \{ , \}, \mu, \alpha)$, the operation $\{ , \}$ is called *Hom-Poisson bracket*. In terms of elements $x, y, z \in A$, the Hom-Leibniz identity says

$$\{xy, \alpha(z)\} = \alpha(x)\{y, z\} + \{x, z\}\alpha(y)$$

where as usual $\mu(x, y)$ is abbreviated to xy . By the antisymmetry of the Hom-Poisson bracket $\{ , \}$, the Hom-Leibniz identity is equivalent to

$$\{\alpha(x), yz\} = \{x, y\}\alpha(z) + \alpha(y)\{x, z\}.$$

We recover Poisson algebras when the twisting map is the identity.

Definition 2.2. A *Hom-Poisson bialgebra* $(A, \mu, \eta, \Delta, \varepsilon, \alpha, \{ , \})$ is a Hom-Poisson algebra $(A, \mu, \{ , \}, \alpha)$ which is also a Hom-bialgebra $(A, \mu, \eta, \Delta, \varepsilon, S, \alpha)$, the two structures being compatible in the sense that $\{ , \}$ is a μ -coderivation,

$$\Delta \circ \{ , \} = (\{ , \} \otimes \mu + \mu \otimes \{ , \}) \circ \Delta^{[2]}.$$

In term of elements, this compatibility condition writes

$$\Delta(\{a, b\}) = \{\Delta(a), \Delta(b)\}$$

with the Hom-Poisson bracket on $A \otimes A$ defined by

$$\{a_1 \otimes a_2, b_1 \otimes b_2\} := \{a_1, b_1\} \otimes a_2 b_2 + a_1 b_1 \otimes \{a_2, b_2\}.$$

We have the same definition for Hom-Poisson Hopf algebras.

Example 2.3. Let $\{x_1, x_2, x_3\}$ be a basis of a 3-dimensional vector space A over \mathbb{K} . The following multiplication μ , skew-symmetric bracket and linear map α on A define a Hom-Poisson algebra over \mathbb{K}^3 :

$$\begin{aligned}\mu(x_1, x_1) &= x_1, & \{x_1, x_2\} &= ax_2 + bx_3, \\ \mu(x_1, x_2) &= \mu(x_2, x_1) = x_3, & \{x_1, x_3\} &= cx_2 + dx_3,\end{aligned}$$

$$\alpha(x_1) = \lambda_1 x_2 + \lambda_2 x_3, \quad \alpha(x_2) = \lambda_3 x_2 + \lambda_4 x_3, \quad \alpha(x_3) = \lambda_5 x_2 + \lambda_6 x_3$$

where $a, b, c, d, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ are parameters in \mathbb{K} .

Theorem 2.4 ([45]). *Let $A = (A, \mu, \{, \})$ be a Poisson algebra, and $\alpha : A \rightarrow A$ be a linear map which is multiplicative for μ and $\{, \}$. Then*

$$A_\alpha = (A, \mu_\alpha = \alpha \circ \mu, \{, \}_\alpha = \alpha \circ \{, \}, \alpha)$$

is a Hom-Poisson algebra.

Proof. Using Theorem (1.5), we already have that (A, μ_α, α) is a commutative Hom-associative algebra and that $(A, \{, \}_\alpha, \alpha)$ is a Hom-Lie algebra. It remains to check the Hom-Leibniz identity

$$\begin{aligned}\{, \}_\alpha \circ (\mu_\alpha \otimes \alpha) &= \alpha \circ \{, \} \circ \alpha^{\otimes 2} \circ (\mu \otimes Id) \\ &= \alpha^2 \circ \{, \} \circ (\mu \otimes Id),\end{aligned}$$

since A is a Poisson algebra

$$\begin{aligned}&= \alpha^2 \circ \mu \circ (Id \otimes \{, \}) + (\{, \} \otimes Id) \circ \tau_{23} \\ &= \alpha \circ \mu \circ \alpha^{\otimes 2} \circ (Id \otimes \{, \}) + (\{, \} \otimes Id) \circ \tau_{23} \\ &= \mu_\alpha \circ (\alpha \otimes \{, \}_\alpha) + (\{, \}_\alpha \otimes \alpha) \circ \tau_{23},\end{aligned}$$

so A_α is a Hom-Poisson algebra. \square

Example 2.5. The Sklyanin Poisson algebra $q_4(\mathcal{E})$ (see [36] for a more detailed definition and properties) is defined on $\mathbb{C}[x_0, x_1, x_2, x_3]$ by a parameter $k \in \mathbb{C}$ with the usual polynomial multiplication, and bracket given by $\{, \}$ where brackets between the coordinate functions are defined as

$$\begin{aligned}\{x_i, x_{i+1}\} &= k^2 x_i x_{i+1} - x_{i+2} x_{i+3}, & i &= 1, 2, 3, 4 \pmod{4}. \\ \{x_i, x_{i+2}\} &= k(x_{i+3}^2 - x_{i+1}^2),\end{aligned}$$

We again search a morphism $\alpha : q_4(\mathcal{E}) \rightarrow q_4(\mathcal{E})$ written as $\alpha = (\alpha_{ij})$ with respect to the basis (x_0, x_1, x_2, x_3) , by solving the coefficients α_{ij} in the equations $\alpha([x_i, x_j]) = [\alpha(x_i), \alpha(x_j)]$ with respect to the basis. For simplicity, we take α diagonal, $\alpha_{ij} = 0$ if $i \neq j$.

We obtain $q_4(\mathcal{E})_\alpha$, Hom-Poisson versions of $q_4(\mathcal{E})$, with α given by

1. $\alpha(x_0) = -\lambda x_0, \alpha(x_1) = i\lambda x_1, \alpha(x_2) = \lambda x_2, \alpha(x_3) = -i\lambda x_3,$
2. $\alpha(x_0) = -\lambda x_0, \alpha(x_1) = -i\lambda x_1, \alpha(x_2) = \lambda x_2, \alpha(x_3) = i\lambda x_3,$
3. $\alpha(x_0) = \lambda x_0, \alpha(x_1) = -\lambda x_1, \alpha(x_2) = \lambda x_2, \alpha(x_3) = -\lambda x_3,$
4. $\alpha = \lambda id,$

with $\lambda \in \mathbb{K}$.

For example, $q_4(\mathcal{E})$ carries a structure of Hom-Poisson algebra, for any $\lambda \in \mathbb{C}$, with the following bracket

$$\begin{aligned} \{x_i, x_{i+1}\} &= -\lambda^2(k^2 x_i x_{i+1} - x_{i+2} x_{i+3}), \\ \{x_i, x_{i+2}\} &= \lambda^2 k(x_{i+3}^2 - x_{i+1}^2), \end{aligned} \quad i = 1, 2, 3, 4 \pmod{4},$$

and linear map

$$\alpha(x_0) = \lambda x_0, \alpha(x_1) = -\lambda x_1, \alpha(x_2) = \lambda x_2, \alpha(x_3) = -\lambda x_3.$$

2.1 Constructing Hom-Poisson algebras from Hom-Lie algebras

Suppose that $(A, [\ , \], \alpha)$ is a finite dimensional Hom-Lie algebra and $\{e_i\}_{1 \leq i \leq n}$ be a basis of A . Set C_{ij}^k be the structure constants of the bracket with respect to the basis, that is $[e_i, e_j] = \sum_{k=1}^n C_{ij}^k e_k$ and α_i^s be the coefficients of the morphism α , that is $\alpha(e_i) = \sum_{s=1}^n \alpha_i^s e_s$.

The skew-symmetry of the bracket and the Hom-Jacobi condition can be written with the structure constants as

$$\begin{aligned} C_{ji}^k &= -C_{ij}^k \quad \text{antisymmetry,} \\ \sum_{1 \leq p, q \leq n} (\alpha_i^p C_{jk}^q + \alpha_j^p C_{ki}^q + \alpha_k^p C_{ij}^q) C_{pq}^s &= 0 \quad \text{Hom-Jacobi identity.} \end{aligned}$$

To construct a Hom-Poisson algebra from a Hom-Lie algebra, we should define a commutative multiplication \cdot which is Hom-associative and a bracket $\{ \ , \ }$ satisfying the Hom-Leibniz identity. Define the bracket $\{ \ , \ }$ as being equal to the bracket $[\ , \]$ on the basis, and extended by the Hom-Leibniz identity.

Set M_{ij}^k be the structure constants for the multiplication, that is $e_i \cdot e_j = \sum_{k=1}^n M_{ij}^k e_k$. By commutativity, $M_{ji}^k = M_{ij}^k$. The Hom-Leibniz identity writes

$$\begin{aligned} 0 &= \{e_i \cdot e_j, \alpha(e_k)\} - \alpha(e_i) \cdot \{e_j, e_k\} - \{e_i, e_k\} \cdot \alpha(e_j) \\ \Leftrightarrow 0 &= \sum_{s=1}^n \underbrace{(M_{ij}^p \alpha_k^q C_{pq}^s - (\alpha_i^p C_{jk}^q + C_{ik}^p \alpha_j^q) M_{pq}^s)}_{S_{ijk s}} e_s \end{aligned}$$

$$\Leftrightarrow 0 = S_{ijks},$$

giving a linear system in the M_{ij}^l of n^4 equations in n^3 unknowns¹.

The Hom-associativity writes

$$\begin{aligned} 0 &= (e_i \cdot e_j) \cdot \alpha(e_k) - \alpha(e_i) \cdot (e_j \cdot e_k) \\ \Leftrightarrow 0 &= \sum_{s=1}^n \underbrace{((M_{ij}^p \alpha_k^q - \alpha_i^p M_{jk}^q) M_{pq}^s)}_{R_{ijks}} e_s \\ \Leftrightarrow 0 &= R_{ijks}, \end{aligned}$$

giving a non linear system in the M_{ij}^l of n^4 equations in n^3 unknowns.

Solving first the equations of Hom-Leibniz and then checking if the solutions satisfy the Hom-associativity equations, we obtain example of Hom-Poisson algebras.

Example 2.6. We consider the 3-dimensional Hom-Lie algebra with basis $\{e_1, e_2, e_3\}$, brackets given by

$$\begin{aligned} [e_1, e_2] &= C_{12}^2 e_2 + C_{12}^3 e_3 \\ [e_1, e_3] &= C_{13}^2 e_2 + C_{13}^3 e_3 \\ [e_2, e_3] &= 0, \end{aligned}$$

and morphism $\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}$ in the basis $\{e_1, e_2, e_3\}$. The only multiplications giving a Hom-Poisson algebra are of the form

$$\begin{aligned} e_1 \cdot e_2 &= \lambda e_2 \\ e_1 \cdot e_3 &= \lambda e_3 \\ e_2 \cdot e_3 &= 0 \\ e_i \cdot e_i &= 0 \quad \text{for } i = 1, 2, 3. \end{aligned}$$

Example 2.7. Other examples of Hom-Poisson algebras of dimension 3 with basis $\{e_1, e_2, e_3\}$ are given by twisting the following Poisson algebra:

$$e_1 \cdot e_1 = e_2 \quad \{e_1, e_3\} = ae_2 + be_3,$$

with all other multiplication and brackets equal to zero.

The morphism α is computed to be multiplicative for the multiplication and the bracket.

<p>With $a \neq 0, b \neq 0$</p> $\begin{aligned} \alpha(e_1) &= e_1 + \alpha_{12}e_2 + \alpha_{13}e_3 \\ \alpha(e_2) &= e_2 \\ \alpha(e_3) &= \alpha_{32}e_2 + \frac{b}{a}\alpha_{32}e_3 \end{aligned}$	<p>With $a \neq 0, b = 0$</p> $\begin{aligned} \alpha(e_1) &= ce_1 + \alpha_{12}e_2 + \alpha_{13}e_3 \\ \alpha(e_2) &= c^2e_2 \\ \alpha(e_3) &= \alpha_{31}e_1 + \alpha_{32}e_2 + \alpha_{33}e_3 \end{aligned}$ <p>where c is a solution (if it exist) of $X^2 - \alpha_{33}X + \alpha_{13}\alpha_{31} = 0$.</p>
---	---

¹actually $\frac{n^2(n+1)}{2}$ unknowns using the commutativity

2.2 Flexibles structures

We recall here some results on flexible structures described in [29] and provide a connection to Hom-Poisson algebras.

Definition 2.8. A Hom-algebra $A = (A, \mu, \alpha)$ is called flexible if for any $x, y \in A$

$$(2.2) \quad \mu(\mu(x, y), \alpha(x)) = \mu(\alpha(x), \mu(y, x)).$$

Remark 2.9. Using the Hom-associator $\mathbf{as}_{\mu, \alpha}$, the condition (2.2) may be written as

$$\mathbf{as}_{\mu, \alpha}(x, y, x) = 0.$$

Lemma 2.10. *Let $A = (A, \mu, \alpha)$ be a Hom-algebra. The following assertions are equivalent*

(1) *A is flexible.*

(2) *For any $x, y \in A$, $\mathbf{as}_{\mu, \alpha}(x, y, x) = 0$.*

(3) *For any $x, y, z \in A$, $\mathbf{as}_{\mu, \alpha}(x, y, z) = -\mathbf{as}_{\mu, \alpha}(z, y, x)$.*

Proof. The equivalence of the first two assertions follows from the definition. The assertion $\mathbf{as}_{\mu, \alpha}(x - z, y, x - z) = 0$ holds by definition and it is equivalent to $\mathbf{as}_{\mu, \alpha}(x, y, z) + \mathbf{as}_{\mu, \alpha}(z, y, x) = 0$ by linearity. \square

Corollary 2.11. *Any Hom-associative algebra is flexible.*

Let $A = (A, \mu, \alpha)$ be a Hom-algebra, where μ is the multiplication and α a homomorphism. We define two new multiplications using μ :

$$\forall x, y \in A \quad x \bullet y = \mu(x, y) + \mu(y, x), \quad \{x, y\} = \mu(x, y) - \mu(y, x).$$

We set $A^+ = (A, \bullet, \alpha)$ and $A^- = (A, \{, \}, \alpha)$.

Proposition 2.12. *A Hom-algebra $A = (A, \mu, \alpha)$ is flexible if and only if*

$$(2.3) \quad \{\alpha(x), y \bullet z\} = \{x, y\} \bullet \alpha(z) + \alpha(y) \bullet \{x, z\}.$$

Proof. Let A be a flexible Hom-algebra. By Lemma (2.10), this is equivalent to $\mathbf{as}_{\mu, \alpha}(x, y, z) + \mathbf{as}_{\mu, \alpha}(z, y, x) = 0$ for any $x, y, z \in A$. This implies

$$(2.4) \quad \mathbf{as}_{\mu, \alpha}(x, y, z) + \mathbf{as}_{\mu, \alpha}(z, y, x) + \mathbf{as}_{\mu, \alpha}(x, z, y) + \mathbf{as}_{\mu, \alpha}(y, z, x) - \mathbf{as}_{\mu, \alpha}(y, x, z) - \mathbf{as}_{\mu, \alpha}(z, x, y) = 0$$

By expansion, the previous relation is equivalent to $\{\alpha(x), y \bullet z\} = \{x, y\} \bullet \alpha(z) + \alpha(y) \bullet \{x, z\}$. Conversely, assume we have the condition (2.3) in Proposition. By setting $x = z$ in (2.4), one gets $\mathbf{as}_{\mu, \alpha}(x, y, x) = 0$. Therefore A is flexible. \square

Hence, we obtain the following connection to Hom-Poisson algebras.

Proposition 2.13. *Let $A = (A, \mu, \alpha)$ be a flexible Hom-algebra which is Hom-associative. Then $(A, \bullet, \{, \}, \alpha)$, where \bullet and $\{, \}$ are the operations defining A^+ and A^- respectively, is a Hom-Poisson algebra.*

2.3 1-operation Poisson algebras

Algebras with one operation were introduced by Loday and studied by Markl and Remm in [33]. The twisted version was studied in [45] where they are called admissible Hom-Poisson algebras.

Definition 2.14. A 1-operation Hom-Poisson algebra is a Hom-algebra (A, \cdot, α) satisfying, for any $x, y, z \in A$,

$$(2.5) \quad 3\mathbf{as}_{\cdot, \alpha}(x, y, z) = (x \cdot z) \cdot \alpha(y) + (y \cdot z) \cdot \alpha(x) - (y \cdot x) \cdot \alpha(z) - (z \cdot x) \cdot \alpha(y).$$

If α is the identity map, A is called a 1-operation Poisson algebra.

We consider a Hom-algebra (A, \cdot, α) . We define two operations $\bullet : A \otimes A \rightarrow A$ and $\{ \cdot, \cdot \} : A \otimes A \rightarrow A$ by

$$(2.6) \quad \forall x, y \in A, \quad x \bullet y = x \cdot y + y \cdot x, \quad \{x, y\} = x \cdot y - y \cdot x.$$

Theorem 2.15. $(A, \bullet, \{ \cdot, \cdot \}, \alpha)$ is a Hom-Poisson algebra if and only if (A, \cdot, α) is a 1-operation Hom-Poisson algebra.

Proof. Suppose that $(A, \bullet, \{ \cdot, \cdot \}, \alpha)$ is a Hom-Poisson algebra. Since

$$(2.7) \quad \forall x, y \in A, \quad x \cdot y = \frac{1}{2}(\{x, y\} + x \bullet y),$$

we have, by expansion,

$$\mathbf{as}_{\cdot, \alpha}(x, y, z) = (x \cdot y) \cdot \alpha(z) - \alpha(x) \cdot (y \cdot z) = \frac{1}{4}\{\alpha(y), \{z, x\}\},$$

and, on the other hand, using that the multiplication \bullet is Hom-associative and commutative, and that $\{ \cdot, \cdot \}$ is a Hom-Lie bracket,

$$(x \cdot z) \cdot \alpha(y) + (y \cdot z) \cdot \alpha(x) - (y \cdot x) \cdot \alpha(z) - (z \cdot x) \cdot \alpha(y) = \frac{3}{4}\{\alpha(y), \{z, x\}\}.$$

We thus have the equation (2.5).

Suppose now that the equation (2.5) is verified. We have to show that \bullet is Hom-associative and that $\{ \cdot, \cdot \}$ is a Hom-Lie bracket.

Using the relation (2.5), we obtain the identities

$$(2.8) \quad \forall x, y, z \in A \quad \mathbf{as}_{\cdot, \alpha}(x, y, z) + \mathbf{as}_{\cdot, \alpha}(y, z, x) + \mathbf{as}_{\cdot, \alpha}(z, x, y) = 0$$

$$(2.9) \quad \forall x, y, z \in A \quad \mathbf{as}_{\cdot, \alpha}(x, y, z) + \mathbf{as}_{\cdot, \alpha}(z, y, x) = 0.$$

This last identity (2.9) shows that (A, \cdot, α) is a Hom-flexible algebra using Lemma 2.10.

We now obtain

$$\begin{aligned} \mathfrak{as}_{\bullet, \alpha}(x, y, z) &= (x \bullet y) \bullet \alpha(z) - \alpha(x) \bullet (y \bullet z) \\ &= \mathfrak{as}_{\cdot, \alpha}(y, z, x) + \mathfrak{as}_{\cdot, \alpha}(x, z, y) - (\mathfrak{as}_{\cdot, \alpha}(z, y, x) + \mathfrak{as}_{\cdot, \alpha}(x, y, z)) \\ &\stackrel{(2.9)}{=} 0. \end{aligned}$$

So the product \bullet is Hom-associative and commutative by definition. Moreover,

$$\begin{aligned} J_{\{ \cdot, \cdot \}, \alpha}(x, y, z) &= \{\alpha(x), \{y, z\}\} + \{\alpha(y), \{z, x\}\} + \{\alpha(z), \{x, y\}\} \\ &= -(\mathfrak{as}_{\cdot, \alpha}(x, y, z) + \mathfrak{as}_{\cdot, \alpha}(y, z, x) + \mathfrak{as}_{\cdot, \alpha}(z, x, y)) + \\ &\quad \mathfrak{as}_{\cdot, \alpha}(x, z, y) + \mathfrak{as}_{\cdot, \alpha}(y, x, z) + \mathfrak{as}_{\cdot, \alpha}(z, y, x) \\ &\stackrel{(2.8)}{=} 0, \end{aligned}$$

so $\{ \cdot, \cdot \}$ is a Hom-Lie bracket. Since A is flexible, Proposition 2.12 leads to the compatibility between \bullet and $\{ \cdot, \cdot \}$,

$$\{\alpha(x), y \bullet z\} = \{x, y\} \bullet \alpha(z) + \alpha(y) \bullet \{x, z\}.$$

So $(A, \bullet, \{ \cdot, \cdot \}, \alpha)$ is a Hom-Poisson algebra. \square

Proposition 2.16. *Let (A, \cdot) be a 1-operation Poisson algebra, and $\alpha : A \rightarrow A$ be a linear map multiplicative for the multiplication \cdot , i.e. $\alpha \circ \cdot = \cdot \circ \alpha^{\otimes 2}$, then $A_\alpha = (A, \cdot_\alpha = \alpha \circ \cdot, \alpha)$ is a 1-operation Hom-Poisson algebra.*

Proof. We have

$$\begin{aligned} 3\mathfrak{as}_{\cdot_\alpha, \alpha}(x, y, z) &= (x \cdot_\alpha y) \cdot_\alpha \alpha(z) - \alpha(x) \cdot_\alpha (y \cdot_\alpha z) \\ &= \alpha(\alpha(x \cdot y) \cdot \alpha(z)) - \alpha(\alpha(x) \cdot \alpha(y \cdot z)) = \alpha^2((x \cdot y) \cdot z - x \cdot (y \cdot z)), \end{aligned}$$

and since \cdot verifies the 1-operation equation,

$$\begin{aligned} 3\mathfrak{as}_{\cdot, \alpha}(x, y, z) &= \alpha^2((x \cdot z) \cdot y + (y \cdot z) \cdot x - (y \cdot x) \cdot z - (z \cdot x) \cdot y) \\ &= \alpha(\alpha(x \cdot z) \cdot \alpha(y)) + \alpha(\alpha(y \cdot z) \cdot \alpha(x)) \\ &\quad - \alpha(\alpha(y \cdot x) \cdot \alpha(z)) - \alpha(\alpha(z \cdot x) \cdot \alpha(y)) \\ &= (x \cdot_\alpha z) \cdot_\alpha (y) + (y \cdot_\alpha z) \cdot_\alpha \alpha(x) - (y \cdot_\alpha x) \cdot_\alpha \alpha(z) - (z \cdot_\alpha x) \cdot_\alpha \alpha(y). \end{aligned}$$

\square

3 Hom-coPoisson structures

Definition 3.1. A *Hom-coPoisson algebra* consists of a cocommutative coassociative Hom-coalgebra $(A, \Delta, \varepsilon, \alpha)$ equipped with a skew-symmetric linear map $\delta : A \rightarrow A \otimes A$, the Hom-coPoisson cobracket, satisfying the following conditions

(i) (Hom-coJacobi identity)

$$(3.1) \quad (Id + \sigma + \sigma^2) \circ (\alpha \otimes \delta) \circ \delta = 0,$$

(ii) (Hom-coLeibniz rule)

$$(3.2) \quad (\Delta \otimes \alpha) \circ \delta = (\alpha \otimes \delta) \circ \Delta + \tau_{23} \circ (\delta \otimes \alpha) \circ \Delta.$$

It is denoted by a tuple $(A, \Delta, \varepsilon, \alpha, \delta)$.

Proposition 3.2. *Let $(A, \Delta, \varepsilon, \alpha, \delta)$ be a Hom-coPoisson algebra and $\beta : A \rightarrow A$ be a Hom-coPoisson algebra morphism. Then $(A, \Delta_\beta = \Delta \circ \beta, \varepsilon, \alpha \circ \beta, \delta_\beta = \delta \circ \beta)$ is a Hom-coPoisson algebra.*

Proof. Theorem 1.9 insures that $(A, \Delta_\beta, \varepsilon, \alpha \circ \beta)$ is a coassociative Hom-coalgebra and Theorem 1.20 that $(A, \delta_\beta, \alpha \circ \beta)$ is a Hom-Lie coalgebra. It remains to show the compatibility condition 3.2. On the left hand side, we have

$$(\Delta_\beta \otimes \alpha \circ \beta) \circ \delta_\beta = (\Delta \circ \beta \otimes \alpha \circ \beta) \circ \delta \circ \beta = (\Delta \otimes \alpha) \circ \beta^2,$$

and the right hand side gives

$$\begin{aligned} & (\alpha \circ \beta \otimes \delta_\beta) \circ \Delta_\beta + \tau_{23} \circ (\delta_\beta \otimes \alpha \circ \beta) \circ \Delta_\beta \\ &= (\alpha \circ \beta \otimes \delta \circ \beta) \circ \Delta \circ \beta + \tau_{23} \circ (\delta \circ \beta \otimes \alpha \circ \beta) \circ \Delta \circ \beta \\ &= [(\alpha \otimes \delta) \circ \Delta + \tau_{23} \circ (\delta \otimes \alpha) \circ \Delta] \circ \beta^2, \end{aligned}$$

which ends the proof. \square

We may state the following Corollaries. Starting from a classical coPoisson algebra, we may construct Hom-coPoisson algebras using coPoisson algebra morphisms. On the other hand a Hom-coPoisson algebra gives rise to infinitely many Hom-coPoisson algebras.

Corollary 3.3.

1. *Let $(A, \Delta, \varepsilon, \delta)$ be a coPoisson algebra and $\beta : A \rightarrow A$ be a coPoisson algebra morphism. Then $(A, \Delta_\beta = \Delta \circ \beta, \varepsilon, \beta, \delta_\beta = \delta \circ \beta)$ is a Hom-coPoisson algebra.*
2. *Let $(A, \Delta, \varepsilon, \alpha, \delta)$ be a Hom-coPoisson algebra. Then for any non negative integer n , we have $(A, \Delta \circ \alpha^n, \varepsilon, \alpha^{n+1}, \delta \circ \alpha^n)$ is a Hom-coPoisson algebra.*

Definition 3.4. A Hom-coPoisson bialgebra $(A, \mu, \eta, \Delta, \varepsilon, \alpha, \delta)$ is a Hom-coPoisson algebra $(A, \Delta, \varepsilon, \alpha, \delta)$ which is also a Hom-bialgebra $(A, \mu, \eta, \Delta, \varepsilon, \alpha)$, the two structures being compatible in the sense that δ is a Δ -derivation,

$$\delta \circ \mu = (\mu \otimes \mu) \circ \tau_{23} \circ (\delta \otimes \Delta + \Delta \otimes \delta).$$

A Hom-coPoisson Hopf algebra $(A, \mu, \eta, \Delta, \varepsilon, S, \alpha, \delta)$ is a Hom-coPoisson bialgebra $(A, \mu, \eta, \Delta, \varepsilon, \alpha, \delta)$ with an antipode S , such that the tuple $(A, \mu, \eta, \Delta, \varepsilon, S, \alpha)$ is a Hom-Hopf algebra.

We extend the connection between Lie bialgebras and coPoisson-Hopf algebras presented in [10] to the Hom setting.

Proposition 3.5. *Let $(\mathfrak{g}, [,], \alpha)$ be a Hom-Lie algebra. If its universal enveloping algebra $U_{HLie}(\mathfrak{g})$ has a Hom-coPoisson structure δ , making it a Hom-coPoisson bialgebra, then $\delta(\mathfrak{g}) \subset \mathfrak{g} \otimes \mathfrak{g}$ and $\delta|_{\mathfrak{g}}$ equips $(\mathfrak{g}, [,], \alpha, \delta, Id)$ with a structure of Hom-Lie bialgebra. Conversely, for a Hom-Lie bialgebra $(\mathfrak{g}, [,], \delta, \alpha)$, the cobracket $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ extends uniquely to a Hom-coPoisson cobracket on $U_{HLie}(\mathfrak{g})$, which makes $U_{HLie}(\mathfrak{g})$ a Hom-coPoisson bialgebra.*

Proof. Let $\delta : U_{HLie}(\mathfrak{g}) \rightarrow U_{HLie}(\mathfrak{g}) \otimes U_{HLie}(\mathfrak{g})$ be a Hom-coPoisson cobracket on $U_{HLie}(\mathfrak{g})$. To show that $\delta(\mathfrak{g}) \subset \mathfrak{g} \otimes \mathfrak{g}$, let $x \in \mathfrak{g}$, and write $\delta(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)}$ where $x^{(1)}, x^{(2)} \in U_{HLie}(\mathfrak{g})$. We may assume that the $x^{(2)}$ are linearly independent. By the Hom-coLeibniz condition (3.2), we have

$$\begin{aligned} \sum_{(x)} \Delta(x^{(1)}) \otimes \alpha(x^{(2)}) &= \alpha(1) \otimes \delta(\alpha(x)) + \alpha(\alpha(x)) \otimes \delta(1) \\ &\quad + \tau_{23} \circ (\delta(1) \otimes \alpha(\alpha(x)) + \delta(\alpha(x)) \otimes \alpha(1)) \end{aligned}$$

since $x \in \mathfrak{g}$ and $\Delta(x) = 1 \otimes \alpha(x) + \alpha(x) \otimes 1$. Moreover, α is a morphism so $\alpha(1) = 1$ and δ is a Δ -derivation so $\delta(1) = 0$, hence

$$\begin{aligned} \sum_{(x)} \Delta(x^{(1)}) \otimes \alpha(x^{(2)}) &= 1 \otimes \delta(\alpha(x)) + \tau_{23} \circ (\delta(\alpha(x)) \otimes 1) \\ &= \sum_{(x)} (1 \otimes \alpha(x^{(1)}) + \alpha(x^{(1)}) \otimes 1) \otimes \alpha(x^{(2)}) \end{aligned}$$

using the multiplicativity $\delta \circ \alpha = \alpha^{\otimes 2} \circ \delta$ of the Hom-coPoisson morphism α . It follows that the $x^{(1)}$ are Hom-primitive elements ($\Delta(x) = 1 \otimes \alpha(x) + \alpha(x) \otimes 1$) of $U_{HLie}(\mathfrak{g})$, hence $\delta(\mathfrak{g}) \subset \mathfrak{g} \otimes U_{HLie}(\mathfrak{g})$. Since δ is skew-symmetric,

$$\delta(\mathfrak{g}) \subset (\mathfrak{g} \otimes U_{HLie}(\mathfrak{g})) \cap (U_{HLie}(\mathfrak{g}) \otimes \mathfrak{g}) = \mathfrak{g} \otimes \mathfrak{g}.$$

To prove the compatibility condition (1.17) for $\delta|_{\mathfrak{g}}$ and the twisting maps α and Id , let $x, y \in \mathfrak{g}$ and compute

$$\begin{aligned} \delta([x, y]) &= \delta(xy - yx) \\ &= \delta(x)\Delta(y) + \Delta(x)\delta(y) - \delta(y)\Delta(x) - \Delta(y)\delta(x) \\ &= [\Delta(x), \delta(y)] - [\Delta(y), \delta(x)] \\ &= [\alpha(x), y^{(1)}] \otimes y^{(2)} + y^{(1)} \otimes [\alpha(x), y^{(2)}] \\ &\quad - [\alpha(y), x^{(1)}] \otimes x^{(2)} - x^{(1)} \otimes [\alpha(y), x^{(2)}] \\ &= ad_{\alpha(x)}(\Delta(y)) - ad_{\alpha(y)}(\Delta(x)). \end{aligned}$$

Conversely, $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ uniquely extends in $\bar{\delta} : U_{HLie}(\mathfrak{g}) \rightarrow U_{HLie}(\mathfrak{g}) \otimes U_{HLie}(\mathfrak{g})$ such that $\bar{\delta}|_{\mathfrak{g}} = \delta$, with the formula

$$\bar{\delta}(xy) = \bar{\delta}(x)\Delta(y) + \Delta(x)\bar{\delta}(y).$$

This gives $U_{HLie}(\mathfrak{g})$ a structure of Hom-coPoisson bialgebra. \square

3.1 Duality

In this section, we extend to Hom-algebras the result stated in [35], that the Hopf dual of a coPoisson Hopf algebra is a Poisson-Hopf algebra.

Definition 3.6. An algebra A over \mathbb{K} is said to be an almost normalizing extension over \mathbb{K} if A is a finitely generated \mathbb{K} -algebra with generators x_1, \dots, x_n satisfying the condition

$$(3.3) \quad x_i x_j - x_j x_i \in \sum_{l=1}^n \mathbb{K} x_l + \mathbb{K}$$

for all i, j .

Lemma 3.7. *Let A be an almost normalizing extension of \mathbb{K} with generators x_1, \dots, x_n . Then A is spanned by all standard monomials*

$$x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}, \quad r_i \in \mathbb{N}$$

together with the unity 1.

Proof. This follows immediately from induction on the degree of monomials. \square

Recall that the bialgebra (resp. Hopf) dual A° of a Hom-bialgebra (resp. Hom-Hopf algebra) A consists of

$$A^\circ = \{f \in A^*, f(I) = 0 \text{ for some cofinite ideal } I \text{ of } A\},$$

where A^* is the dual vector space of A .

Theorem 3.8. *Let A be a Hom-coPoisson bialgebra (resp. Hopf algebra) with Poisson co-bracket δ and twisting multiplicative map α . If A is an almost normalizing extension over \mathbb{K} , then the bialgebra (resp. Hopf) dual A° is a Hom-Poisson bialgebra (resp. Hopf algebra) with twisting map α° and bracket*

$$(3.4) \quad \{f, g\}(x) = \langle \delta(x), f \otimes g \rangle, \quad x \in A$$

for any $f, g \in A^\circ$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between the vector space $A \otimes A$ and its dual vector space.

Proof. The proof is almost the same as in [35]. We do not reproduce here the first step showing that the bracket (3.4) is well-defined, it uses the fact that A is an almost normalizing extension over \mathbb{K} .

The skew-symmetry follows from $\tau_{12} \circ \delta = -\delta$, we have

$$\begin{aligned} \{g, f\}(x) &= \langle \delta(x), g \otimes f \rangle = \langle \tau_{12} \circ \delta, f \otimes g \rangle \\ &= -\langle \delta(x), f \otimes g \rangle = -\{f, g\}(x), \end{aligned}$$

for all $x \in A$.

The equation (3.4) satisfies the Hom-Leibniz rule: since

$$(3.5) \quad \{fg, \alpha^\circ(h)\}(x) = \langle (\Delta \otimes \alpha) \circ \delta(x), f \otimes g \otimes h \rangle$$

and

$$\begin{aligned} & (\alpha^\circ(f)\{g, h\} + \{f, h\}\alpha^\circ(g))(x) \\ &= \langle (\alpha \otimes \delta) \circ \Delta(x), f \otimes g \otimes h \rangle + \langle \tau_{23} \circ (\delta \otimes \alpha) \circ \Delta(x), f \otimes g \otimes h \rangle \end{aligned}$$

for $x \in A$ and $f, g, h \in A^\circ$, it is enough to show that

$$(3.6) \quad (\Delta \otimes \alpha) \circ \delta = (\alpha \otimes \delta) \circ \Delta + \tau_{23} \circ (\delta \otimes \alpha) \circ \Delta,$$

but this is just the Hom-coLeibniz rule for δ .

The equation (3.4) satisfies the Hom-Jacobi identity: we have

$$\begin{aligned} \{\alpha^\circ(f), \{g, h\}\}(x) &= \langle (\alpha \otimes \delta) \circ \delta(x), f \otimes g \otimes h \rangle, \\ \{\alpha^\circ(g), \{h, f\}\}(x) &= \langle \sigma \circ (\alpha \otimes \delta) \circ \delta(x), f \otimes g \otimes h \rangle, \\ \{\alpha^\circ(h), \{f, g\}\}(x) &= \langle \sigma^2 \circ (\alpha \otimes \delta) \circ \delta(x), f \otimes g \otimes h \rangle, \end{aligned}$$

for $x \in A$ and $f, g, h \in A^\circ$. Hence (3.4) satisfies the Hom-Jacobi identity if and only if δ satisfies

$$(Id + \sigma + \sigma^2) \circ (\alpha \otimes \delta) \circ \delta = 0,$$

which is just the Hom-coJacobi identity of δ .

The bracket defined by (3.4) satisfies the compatibility condition with the comultiplication of the Hom-bialgebra (resp. Hopf algebra), it is a μ -coderivation: since δ is a Δ -derivation, we have for $f, g \in A^\circ$

$$\begin{aligned} \Delta(\{f, g\})(x \otimes y) &= \{f, g\}(xy) = \langle \delta(xy), f \otimes g \rangle \\ &= \langle \delta(x)\Delta(y), f \otimes g \rangle + \langle \Delta(x)\delta(y), f \otimes g \rangle \\ &= \langle \delta(x), f_1 \otimes g_1 \rangle \langle \Delta(y), f_2 \otimes g_2 \rangle \\ &\quad + \langle \Delta(x), f_1 \otimes g_2 \rangle \langle \delta(y), f_2 \otimes g_2 \rangle \\ &= \{f_1, g_1\}(x)(f_2 g_2)(y) + (f_1 g_1)(x)\{f_2, g_2\}(y) \\ &= \{\Delta(f), \Delta(g)\}(x \otimes y). \end{aligned}$$

Finally, the bracket defined by (3.4) equips A° with the structure of a Hom-Poisson bialgebra (resp. Hopf algebra), the twisting map being α° . \square

Let $(\mathfrak{g}, [,], \delta, \alpha)$ be a Hom-Lie bialgebra, $U_{HLie}(\mathfrak{g})$ the universal enveloping Hom-bialgebra of \mathfrak{g} with comultiplication Δ . The cobracket $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ is extended uniquely to a Δ -derivation $\bar{\delta} : U_{HLie}(\mathfrak{g}) \rightarrow U_{HLie}(\mathfrak{g}) \otimes U_{HLie}(\mathfrak{g})$ such that $\bar{\delta}|_{\mathfrak{g}} = \delta$ and $\bar{\delta}(xy) = \bar{\delta}(x)\Delta(y) + \Delta(x)\bar{\delta}(y)$. Then $U_{HLie}(\mathfrak{g})$ is a Hom-coPoisson bialgebra with cobracket $\bar{\delta}$.

Corollary 3.9. *Let $(\mathfrak{g}, [,], \delta, \alpha)$ be a finite dimensional Hom-Lie bialgebra. Then the dual $U_{HLie}(\mathfrak{g})^\circ$ of the universal enveloping Hom-bialgebra $U_{HLie}(\mathfrak{g})$ is a Hom-Poisson bialgebra with Poisson bracket*

$$\{f, g\}(x) = \langle \bar{\delta}(x), f \otimes g \rangle, \quad x \in U_{HLie}(\mathfrak{g})$$

for $f, g \in U_{HLie}(\mathfrak{g})^\circ$.

Proof. Let $\{x_1, \dots, x_n\}$ be a basis of \mathfrak{g} . Then $U_{HLie}(\mathfrak{g})$ is an almost normalizing extension over \mathbb{K} with generators x_1, \dots, x_n . Thus the result follows from Theorem 3.8. \square

4 Deformation theory and Quantization

Deformation is one of the oldest techniques used by mathematicians and physicists. The first instances of the so-called deformation theory were given by Kodaira and Spencer for complex structures and by Gerstenhaber for associative algebras [14]. The Lie algebras case was studied by Nijenhuis and Richardson [34] and the deformation theory for bialgebras and Hopf algebras were developed later by Gerstenhaber and Schack [15]. The main and popular tool is the power series ring or more generally any commutative algebras. By standard facts of deformation theory, the infinitesimal deformations of an algebra of a given type are parametrized by a second cohomology of the algebra. More generally, it is stated that deformations are controlled by a suitable cohomology. A modern approach, essentially due to Quillen, Deligne, Drinfeld, and Kontsevich, is that, in characteristic zero, every deformation problem is controlled by a differential graded Lie algebra, via solutions of Maurer-Cartan equation modulo gauge equivalence.

Some mathematical formulations of quantization are based on the algebra of observables and consist in replacing the classical algebra of observables (typically complex-valued smooth functions on a Poisson manifold) by a noncommutative one constructed by means of an algebraic formal deformation of the classical algebra. The so-called deformation quantization problem was introduced in the seminal paper [4] by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer (1978).

In 1997, Kontsevich solved a longstanding problem in mathematical physics, that is every Poisson manifold admits formal quantization which is canonical up to a certain equivalence.

4.1 Formal deformation of Hom-associative algebras

In [31] the formal deformation theory is extended to Hom-associative and Hom-Lie algebras and a suitable cohomology is provided, see also [2]. The usual results involving deformation first order element and second order cohomology groups extends in the Hom case. We describe briefly the results in this section. We

consider deformations of Hom-associative algebras where the twist map remains unchanged.

Let $A = (A, \mu_0, \alpha)$ be a Hom-associative algebra. Let $\mathbb{K}[[t]]$ be the power series ring in one variable t and coefficients in \mathbb{K} and $A[[t]]$ be the set of formal power series whose coefficients are elements of A , ($A[[t]]$ is obtained by extending the coefficients domain of A from \mathbb{K} to $\mathbb{K}[[t]]$). Then $A[[t]]$ is a $\mathbb{K}[[t]]$ -module. When A is finite-dimensional, we have $A[[t]] = A \otimes \mathbb{K}[[t]]$. Note that A is a submodule of $A[[t]]$.

Definition 4.1. Let $A = (A, \mu_0, \alpha)$ be a Hom-associative algebra. A formal Hom-associative deformation of A is given by a $\mathbb{K}[[t]]$ -bilinear map $\mu_t : A[[t]] \otimes A[[t]] \rightarrow A[[t]]$ of the form

$$(4.1) \quad \mu_t = \sum_{i \geq 0} \mu_i t^i$$

where each μ_i is a \mathbb{K} -bilinear map $\mu_i : A \otimes A \rightarrow A$ (extended to be $\mathbb{K}[[t]]$ -bilinear) such that holds the following formal Hom-associativity condition:

$$(4.2) \quad \mathbf{as}_{\mu_t, \alpha} = \mu_t \circ (\mu_t \otimes \alpha - \alpha \otimes \mu_t) = 0.$$

If $\alpha = Id$ the definition reduces to formal deformations of an associative algebra.

The equation (4.2) can be written

$$(4.3) \quad \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} (\mu_i(\alpha(x), \mu_j(y, z)) - \mu_i(\mu_j(x, y, \alpha(z)))) t^{i+j} = 0.$$

Introducing the following notation

$$(x, y, z) \mapsto \mu_i \circ_{\alpha} \mu_j(x, y, z) := \mu_i(\alpha(x), \mu_j(y, z)) - \mu_i(\mu_j(x, y, \alpha(z))),$$

the deformation equation may be written as follows

$$(4.4) \quad \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} (\mu_i \circ_{\alpha} \mu_j) t^{i+j} = 0 \quad \text{or} \quad \sum_{s \in \mathbb{N}} t^s \sum_{i=0}^s \mu_i \circ_{\alpha} \mu_{s-i} = 0.$$

It is equivalent to the infinite system: $\sum_{i=0}^s \mu_i \circ_{\alpha} \mu_{s-i} = 0$, $s = 0, 1, \dots$.

The A -valued Hochschild Type cohomology of Hom-associative algebras initiated in [31] and extended in [2] suits and leads to the following cohomological interpretation:

1. There is a natural bijection between $H^2(A, A)$ and the set of equivalence classes of deformation (mod t^2) of A .
2. If $H^2(A, A) = 0$ then every deformation of A is trivial.

The fact that the antisymmetrization of the first order element of a deformation of an associative algebra defines a Poisson bracket remains true in the Hom setting. More precisely, we have the following theorem.

Theorem 4.2 ([31]). *Let $A = (A, \mu_0, \alpha)$ be a commutative Hom-associative algebra and $A_t = (A, \mu_t, \alpha)$ be a deformation of A . Consider the bracket defined for $x, y \in A$ by $\{x, y\} = \mu_1(x, y) - \mu_1(y, x)$ where μ_1 is the first order element of the deformation μ_t . Then $(A, \mu_0, \{, \}, \alpha)$ is a Hom-Poisson algebra.*

The proof is mainly computational, it leans on the properties of the α -associators, and on rewriting the deformation equations in terms of coboundary operators.

4.2 Deformations of Hom-coalgebras and Hom-Bialgebras

The formal deformation theory for bialgebras and Hopf algebras was introduced in [15]. It is extended to Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras in [12], where a suitable cohomology is obtained and the classical results are extended to Hom-setting.

Definition 4.3. Let (A, Δ, α) be a Hom-coalgebra. A formal Hom-coalgebra deformation of A is given by a linear map $\Delta_t : A[[t]] \rightarrow A[[t]] \otimes A[[t]]$ such that $\Delta_t = \sum_{i \geq 0} \Delta_i t^i$ where each Δ_i is a linear map $\Delta_i : A \rightarrow A \otimes A$ (extended to be $\mathbb{K}[[t]]$ -linear) such that holds the following formal Hom-coassociativity condition:

$$(4.5) \quad (\Delta_t \otimes \alpha - \alpha \otimes \Delta_t) \circ \Delta_t = 0.$$

Definition 4.4. Let (A, μ, Δ, α) be a Hom-bialgebra. A formal Hom-bialgebra deformation of A is given by linear maps $\mu_t : A[[t]] \otimes A[[t]] \rightarrow A[[t]]$ and $\Delta_t : A[[t]] \rightarrow A[[t]] \otimes A[[t]]$ such that

1. $(A[[t]], \mu_t, \alpha)$ is a Hom-associative algebra,
2. $(A[[t]], \Delta_t, \alpha)$ is a Hom-coassociative coalgebra,
3. The multiplication and the comultiplication are compatible, that is

$$\Delta_t \circ \mu_t = \mu_t \otimes \mu_t \circ \tau_{23} \circ (\Delta_t \otimes \Delta_t).$$

It is shown in [12] that deformations are controlled by Hochschild type cohomology and any deformation of unital Hom-associative algebra (resp. counital Hom-coassociative colgebra) is equivalent to unital Hom-associative algebra (resp. counital Hom-coassociative colgebra). Furthermore, any deformation of a Hom-Hopf algebra as a Hom-bialgebra is automatically a Hom-Hopf algebra.

In a similar way as for Hom-associative algebra, we have:

Theorem 4.5. *Let $A = (A, \Delta_0, \alpha)$ be a cocommutative Hom-coassociative coalgebra and $A_t = (A, \Delta_t, \alpha)$ be a deformation of A . Consider the cobracket defined for $x \in A$ by $\delta(x) = \Delta_1(x) - \Delta_1^{op}(x)$ where Δ_1 is the first order element of the deformation Δ_t . Then $(A, \Delta_0, \delta, \alpha)$ is a Hom-coPoisson algebra.*

4.3 Quantization and Twisting star-products

The deformation quantization problem in the Hom-setting is stated as follows: given a Hom-Poisson algebra (resp. Hom-coPoisson algebra), find a formal deformation of a commutative Hom-associative algebra (resp. cocommutative Hom-coassociative coalgebra) such that the first order element of the deformation defines the Hom-Poisson algebra (resp. Hom-coPoisson algebra). In the classical case the Hom-Poisson structure is called the quasi-classical limit and the deformation is the star-product. This point of view initiated in [4] attempts to view the quantum mechanics as a deformation of the classical mechanics, the Lorentz group is a deformation of the Galilee group.

Let $(A, \cdot, \{, \}, \alpha)$ be a commutative Hom-associative algebra endowed with a Hom-Poisson bracket $\{, \}$.

Definition 4.6. A \star -product on A is a one parameter formal deformation defined on A by

$$f \star_t g = \sum_{r=0}^{\infty} t^r \mu_r(f, g)$$

such that

1. The \star -product in $A[[t]]$ is Hom-associative, that is

$$\forall r \in \mathbb{N}, \quad \sum_{i=0}^r (\mu_i(\mu_{r-i}(f, g), \alpha(h)) - (\mu_i(\alpha(f), \mu_{r-i}(g, h))) = 0,$$

2. $\mu_0(f, g) = f \cdot g$
3. $\mu_1(f, g) - \mu_1(g, f) = \{f, g\}$
4. $\mu_r(f, 1) = \mu_r(1, f) = 0 \quad \forall r > 0$

Remark 4.7.

- The condition (2) shows that $[f, g] := \frac{1}{2t}(f \star_t g - g \star_t f)$ is a deformation of the Hom-Lie structure $\{, \}$.
- The condition $\mu_1(f, g) - \mu_1(g, f) = \{f, g\}$ expresses the correspondence between the deformation and the Hom-Poisson structure

$$\frac{f \star_t g - g \star_t f}{t} \Big|_{t=0} = \{f, g\}.$$

Similarly we set the dual version of quantization problem as follows.

Let A be cocommutative Hom-coPoisson bialgebra (resp. Hopf algebra) and let δ be its Poisson cobracket. A quantization of A is a Hom-bialgebra (resp. Hom-Hopf algebra) deformation A_t of A such that

$$\delta(x) = \frac{\Delta_t(a) - \Delta_t^{op}(a)}{t} \pmod{t},$$

where $x \in A$ and a is any element of $A[[t]]$ such that $x = a \pmod{t}$.

Theorem 4.8. *Let (A, \star) be an associative deformation of an associative algebra (A, μ_0) , with $\star = \mu_0 + t\mu_1 + t^2\mu_2 + \dots$*

Let $\alpha : A \rightarrow A$ be a morphism such that for all $i \in \mathbb{N}$, $\alpha \circ \mu_i = \mu_i \circ \alpha^{\otimes 2}$. Then $(A, \star_\alpha = \alpha \circ \star, \alpha)$ is a Hom-associative deformation of A .

Proof. Since for all $i \in \mathbb{N}$, $\alpha \circ \mu_i = \mu_i \circ \alpha^{\otimes 2}$, we also have $\alpha \circ \star = \star \circ \alpha^{\otimes 2}$. For $f, g, h \in (A, \star_\alpha, \alpha)$, we get

$$(4.6) \quad \begin{aligned} (f \star_\alpha g) \star_\alpha \alpha(h) &= \alpha(\alpha(f) \star \alpha(g)) \star \alpha(\alpha(h)) = \alpha((\alpha(f) \star \alpha(g)) \star \alpha(h)) \\ &= \alpha(\alpha(f) \star (\alpha(g) \star \alpha(h))) = \alpha(\alpha(f)) \star \alpha(\alpha(g) \star \alpha(h)) = \alpha(f) \star_\alpha (g \star_\alpha h), \end{aligned}$$

the passage from the first line to the second is due to the associativity of \star . This shows that the product \star_α is Hom-associative. \square

4.4 Moyal-Weyl Hom-associative algebra

In the following, we twist the Moyal-Weyl product. It is the associative \star -product corresponding to the deformation of the Poisson phase-space bracket, one of the first examples of Kontsevich formal deformation [24].

We consider the Poisson algebra of polynomials of two variables $(\mathbb{R}[x, y], \cdot, \{, \})$ where the Poisson bracket of two polynomials is given by $\{f, g\} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}$.

The associated associative algebra is $(\mathbb{R}[x, y], \star)$ where the star-product is given by the Moyal-Weyl formula

$$(4.7) \quad f \star g = \sum_{n \in \mathbb{N}} \frac{\partial^n f}{\partial x^n} \frac{\partial^n g}{\partial y^n} \frac{t^n}{n!} = \sum_{n \in \mathbb{N}} \mu_n(f, g) t^n,$$

where $\mu_n(f, g) = \frac{1}{n!} \frac{\partial^n f}{\partial x^n} \frac{\partial^n g}{\partial y^n}$.

Proposition 4.9. *A morphism $\alpha : \mathbb{R}[x, y] \rightarrow \mathbb{R}[x, y]$ satisfying $\alpha \circ \mu_i = \mu_i \circ \alpha^{\otimes 2}$ for all $i \in \mathbb{N}$ which gives $(\mathbb{R}[x, y], \star_\alpha = \alpha \star, \alpha)$ a structure of Hom-associative algebra is of the form*

$$(4.8) \quad \alpha(x) = ax + b \text{ and } \alpha(y) = \frac{1}{a}y + c \quad \text{where } a, b, c \in \mathbb{R}, a \neq 0.$$

Proof. Let $\alpha : \mathbb{R}[x, y] \rightarrow \mathbb{R}[x, y]$ be a morphism such that for all $i \in \mathbb{N}$, $\alpha\mu_i = \mu_i\alpha^{\otimes 2}$. In particular, for $i = 0$,

$$(4.9) \quad \alpha(fg) = \alpha(f)\alpha(g),$$

which shows that α is multiplicative, so it is sufficient to define it on x and y . For $i = 1$, we get

$$(4.10) \quad \alpha\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}\right) = \frac{\partial\alpha(f)}{\partial x}\frac{\partial\alpha(g)}{\partial y},$$

which implies that $\alpha(\{f, g\}) = \{\alpha(f), \alpha(g)\}$.

We set $P_1(x, y) := \alpha(x)$ and $P_2 := \alpha(y)$. For $f(x, y) = x$ and $g(x, y) = y$, the equation (4.10) gives

$$1 = \alpha(1) = \frac{\partial P_1}{\partial x}\frac{\partial P_2}{\partial y},$$

so we must have $P_1(x, y) = ax + Q_1(y)$ and $P_2(x, y) = \frac{1}{a}y + Q_2(x)$ with $a \in \mathbb{R} \setminus \{0\}$ and $Q_1, Q_2 \in \mathbb{R}[x, y]$. For $f(x, y) = x$ and $g(x, y) = y$, the equation (4.10) gives

$$0 = \alpha(0) = \frac{\partial P_2}{\partial x}\frac{\partial P_1}{\partial y} = Q'_{2,x}Q'_{1,y}.$$

So we can suppose that $Q_1(y) = b$ is constant. Reporting in the equation (4.10), with $f(x, y) = g(x, y) = y$, we find

$$0 = \alpha(0) = \frac{\partial P_2}{\partial x}\frac{\partial P_2}{\partial y} = Q'_{2,x}\frac{1}{a},$$

so $Q_2(x) = c$ is constant. It remains to verify that for $i > 1$, $\alpha\mu_i = \mu_i\alpha^{\otimes 2}$ *i.e.* for $f, g \in \mathbb{R}[x, y]$, $\alpha\left(\frac{\partial^i f}{\partial x^i}\frac{\partial^i g}{\partial y^i}\frac{1}{i!}\right) = \frac{\partial^i\alpha(f)}{\partial x^i}\frac{\partial^i\alpha(g)}{\partial y^i}\frac{1}{i!}$. By multiplicativity of α , the only non trivial case is $f(x, y) = x^n$ and $g(x, y) = y^m$. We have

$$(4.11) \quad \begin{aligned} \alpha\left(\frac{\partial^i f}{\partial x^i}\frac{\partial^i g}{\partial y^i}\frac{1}{i!}\right) &= \alpha\left(i!\binom{n}{i}x^{n-i}i!\binom{m}{i}y^{m-i}\frac{1}{i!}\right) \\ &= i!\binom{n}{i}i!\binom{m}{i}(ax+b)^{n-i}\left(\frac{1}{a}y+c\right)^{m-i}\frac{1}{i!} \\ &= i!\binom{n}{i}i!\binom{m}{i}(ax+b)^{n-i}a^i\left(\frac{1}{a}y+c\right)^{m-i}\left(\frac{1}{a}\right)^i\frac{1}{i!} = \frac{\partial^i(ax+b)^n}{\partial x^i}\frac{\partial^i\left(\frac{1}{a}y+c\right)^m}{\partial y^i}\frac{1}{i!} \\ &= \frac{\partial^i\alpha(f)}{\partial x^i}\frac{\partial^i\alpha(g)}{\partial y^i}\frac{1}{i!}. \end{aligned}$$

□

The Hom-algebra $(\mathbb{R}[x, y], \star_\alpha, \alpha)$ is Hom-associative and not associative if $\alpha \neq Id$. Indeed, for $f(x, y) = 1$, $g(x, y) = y$ and $h(x, y) = x$, we have

$$\begin{aligned} (f \star_\alpha g) \star_\alpha h &= \alpha(\alpha(f) \star \alpha(g)) \star \alpha(h) \\ &= \alpha\left(1 \star \frac{1}{a}y + c\right) \star (ax + b) = \alpha\left(\frac{1}{a}y + c\right) \star (ax + b), \end{aligned}$$

and

$$\begin{aligned} f \star_\alpha (g \star_\alpha h) &= \alpha(\alpha(f)) \star \alpha(\alpha(g) \star \alpha(h)) \\ &= 1 \star \alpha\left(\left(\frac{1}{a}y + c\right) \star (ax + b)\right) = \alpha\left(\frac{1}{a}y + c\right) \star \alpha(ax + b) \end{aligned}$$

which are different in general.

More generally, we can consider the Poisson algebra of polynomials of $n \geq 3$ variables $(\mathbb{R}[x_1, \dots, x_n], \cdot, \{, \})$ where the Poisson bracket of two polynomials is given by $\{f, g\} = \sum_{1 \leq i, j \leq n} \tau_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$, with $\tau = (\tau_{ij})$ an antisymmetric $n \times n$ real matrix.

The associated associative algebra is $(\mathbb{R}[x_1, \dots, x_n], \star)$ where the star-product is given by the Moyal-Weyl formula

$$(4.12) \quad f \star g = \sum_{n \in \mathbb{N}} \sum_{1 \leq i_1, j_1, \dots, i_n, j_n \leq n} \sigma_{i_1 j_1} \cdots \sigma_{i_n j_n} \frac{\partial^n f}{\partial x_{i_1} \cdots \partial x_{i_n}} \frac{\partial^n g}{\partial x_{j_1} \cdots \partial x_{j_n}} \frac{t^n}{n!},$$

where $\sigma = (\sigma_{ij})$ is the matrix whose antisymmetrization is τ .

$$\text{Set } n > 2 \text{ and } \mu_n = \frac{\partial^n f}{\partial x_{i_1} \cdots \partial x_{i_n}} \frac{\partial^n g}{\partial x_{j_1} \cdots \partial x_{j_n}}.$$

Proposition 4.10. *The only morphisms $\alpha : \mathbb{R}[x_1, \dots, x_n] \rightarrow \mathbb{R}[x_1, \dots, x_n]$ satisfying $\alpha \circ \mu_i = \mu_i \circ \alpha^{\otimes 2}$ for all $i \in \mathbb{N}$ which give $(\mathbb{R}[x_1, \dots, x_n], \star_\alpha = \alpha \star, \alpha)$ a structure of Hom-associative algebra are of the form*

$$(4.13) \quad \forall 1 \leq i \leq n, \alpha(x_i) = x_i + b_i \quad \text{or} \quad \forall 1 \leq i \leq n, \alpha(x_i) = -x_i + b_i \quad \text{where } b_i \in \mathbb{R}.$$

Proof. The proof is similar to the case with two variables, we get $\alpha(x_i) = a_i x_i + b_i$, except that this time, $a_i a_j = 1$ for all $i \neq j$, which gives the two cases of the proposition. □

4.5 Moyal-Weyl Hom-Poisson algebra

We consider the Poisson algebra of polynomials of two variables $(\mathbb{R}[x, y], \cdot, \{, \})$ where the Poisson bracket of two polynomials is given by $\{f, g\} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}$.

Proposition 4.11. *A morphism $\alpha : \mathbb{R}[x, y] \rightarrow \mathbb{R}[x, y]$ which gives $(\mathbb{R}[x, y], \cdot, \alpha = \alpha \circ \cdot, \{, \}_\alpha = \alpha \circ \{, \}, \alpha)$ a structure of Hom-Poisson algebra satisfies the equation*

$$(4.14) \quad 1 = \frac{\partial \alpha(x)}{\partial x} \frac{\partial \alpha(y)}{\partial y} - \frac{\partial \alpha(x)}{\partial y} \frac{\partial \alpha(y)}{\partial x}.$$

Proof. Since α is a morphism of Poisson algebra, it satisfies, for all $f, g \in \mathbb{R}[x, y]$

$$\begin{aligned} \alpha(f \cdot g) &= \alpha(f) \cdot \alpha(g) \\ \alpha(\{f, g\}) &= \{\alpha(f), \alpha(g)\}. \end{aligned}$$

The first equation shows that it is sufficient to define α on x and y . For the second equation, we can suppose by linearity that $f(x, y) = x^k y^l$ and $g(x, y) = x^p y^q$. It then rewrites

$$\begin{aligned} &kq\alpha(x)^{k-1}\alpha(y)^l\alpha(x)^p\alpha(y)^{q-1} - pl\alpha(x)^k\alpha(y)^{l-1}\alpha(x)^{p-1}\alpha(y)^q \\ &= \frac{\partial(\alpha(x)^k\alpha(y)^l)}{\partial x} \frac{\partial(\alpha(x)^p\alpha(y)^q)}{\partial y} - \frac{\partial(\alpha(x)^k\alpha(y)^l)}{\partial y} \frac{\partial(\alpha(x)^p\alpha(y)^q)}{\partial x} \end{aligned}$$

which simplifies in

$$1 = \frac{\partial \alpha(x)}{\partial x} \frac{\partial \alpha(y)}{\partial y} - \frac{\partial \alpha(x)}{\partial y} \frac{\partial \alpha(y)}{\partial x}.$$

□

Example 4.12. We give some examples for the morphism α . We set

$$\begin{aligned} \alpha(x) &= \Gamma_1(x, y) = \sum_{0 \leq i, j \leq d} a_{ij} x^i y^j \\ \alpha(y) &= \Gamma_2(x, y) = \sum_{0 \leq i, j \leq d} b_{ij} x^i y^j \end{aligned}$$

with $\Gamma_1, \Gamma_2 \in \mathbb{R}[x, y]$, and d the bigger degree for each variable. Since the equation (4.14) only contains derivatives of Γ_1 and Γ_2 , without loss of generality we assume $a_{00} = b_{00} = 0$.

Degree 1

$$\begin{aligned} \Gamma_1(x, y) &= a_{10}x + a_{01}y \\ \Gamma_2(x, y) &= b_{10}x + b_{01}y \end{aligned}$$

The equation (4.14) becomes

$$(4.15) \quad 1 = a_{10}b_{01} - a_{01}b_{10}.$$

The polynomials Γ_1, Γ_2 are of one of the following form

- (i) $\Gamma_1(x, y) = a_{10}x + a_{01}y$, $\Gamma_2(x, y) = -\frac{1}{a_{10}}x$ with $a_{10} \neq 0$,
(ii) $\Gamma_1(x, y) = \frac{1+a_{01}b_{10}}{b_{01}}x + a_{01}y$, $\Gamma_2(x, y) = b_{10}x + b_{01}y$ with $b_{01} \neq 0$.

Degree 2 For simplicity, we only take one of the polynomials of degree two.

$$\begin{aligned}\Gamma_1(x, y) &= a_{10}x + a_{01}y \\ \Gamma_2(x, y) &= b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{20}y^2\end{aligned}$$

Arranging the equation (4.14) by degree, we obtain the following system of equations.

$$\begin{cases} 1 = a_{10}b_{01} - a_{01}b_{10} \\ 0 = 2a_{10}b_{02} - a_{01}b_{11} \\ 0 = a_{10}b_{11} - 2a_{01}b_{20} \end{cases}$$

The polynomials Γ_1, Γ_2 are of one of the following form

- (i) $\Gamma_1(x, y) = \frac{1+a_{01}b_{10}}{b_{01}}x + a_{01}y$, $\Gamma_2(x, y) = b_{10}x + b_{01}y$
(ii) $\Gamma_1(x, y) = \frac{1}{b_{01}}x$, $\Gamma_2(x, y) = b_{20}x^2 + b_{10}x + b_{01}y$
(iii) $\Gamma_1(x, y) = a_{10}x + \frac{2a_{10}b_{02}}{b_{11}}y$, $\Gamma_2(x, y) = \frac{1}{a_{01}}y + \frac{b_{11}^2}{4b_{02}}x^2 + b_{11}xy + b_{02}y^2$
(iv) $\Gamma_1(x, y) = a_{10}x + \frac{2a_{10}b_{02}}{b_{11}}y$, $\Gamma_2(x, y) = b_{10}x + \frac{2a_{10}b_{02}b_{10}+b_{11}}{a_{10}b_{11}}y + \frac{b_{11}^2}{4b_{02}}x^2 + b_{11}xy + b_{02}y^2$

We now want to deform the classical Moyal-Weyl star-product on $\mathbb{R}[x, y]$ using morphisms α previously found. For $P, Q \in \mathbb{R}[x, y]$ we define

$$(4.16) \quad P \star_\alpha Q = \sum_{n \geq 0} \mu_{n\alpha}(P, Q)t^n \quad \text{with} \quad \mu_{n\alpha}(P, Q) = \frac{\partial\alpha(P)}{\partial x_n} \frac{\partial\alpha(Q)}{\partial y_n} \frac{1}{n!}.$$

The Hom-associator writes

$$(4.17) \quad \begin{aligned} \text{as}_{\star_\alpha, \alpha}(P, Q, R) &= \sum_{p \geq 1} \sum_{n=0}^p (\mu_{p-n\alpha}(\mu_{n\alpha}(P, Q), \alpha(R)) - \mu_{p-n\alpha}(\alpha(P), \mu_{n\alpha}(Q, R)))t^p \\ &= \sum_{p \geq 1} \sum_{n=0}^p \frac{1}{n!(p-n)!} \left(\frac{\partial\alpha \left(\frac{\partial\alpha(P)}{\partial x^n} \frac{\partial\alpha(Q)}{\partial y^n} \right)}{\partial x^{p-n}} \frac{\partial\alpha^2(R)}{\partial y^{p-n}} - \frac{\partial\alpha^2(P)}{\partial x^{p-n}} \frac{\partial\alpha \left(\frac{\partial\alpha(Q)}{\partial x^n} \frac{\partial\alpha(R)}{\partial y^n} \right)}{\partial y^{p-n}} \right), \end{aligned}$$

since the multiplication \cdot_α is Hom-associative, the constant term vanishes.

Trying to make the coefficient in t^n vanish for particular polynomials P, Q, R , we obtain conditions on the morphism α .

4.5.1 Degree 1 case

In degree 1, the solutions of (4.15) are of the form (i) or (ii).

In the first case (i), the morphism α satisfies

$$\begin{aligned}\alpha(x) &= \Gamma_1(x, y) = a_{10}x + a_{01}y, \\ \alpha(y) &= \Gamma_2(x, y) = -\frac{1}{a_{01}}x,\end{aligned}$$

with $a_{10} \neq 0$.

For the particular polynomials $P(x, y) = x, Q(x, y) = y = R(x, y)$, we have

$$\mathfrak{as}_{\star_\alpha, \alpha}(P, Q, R) = y \neq 0$$

so $\mathfrak{as}_{\star_\alpha, \alpha} \neq 0$ and this morphism α does not give a deformation of the Moyal-Weyl star-product.

In the second case (ii) the morphism α satisfies

$$\begin{aligned}\alpha(x) &= \Gamma_1(x, y) = \frac{1 + a_{01}b_{10}}{b_{01}}x + a_{01}y, \\ \alpha(y) &= \Gamma_2(x, y) = b_{10}x + b_{01}y,\end{aligned}$$

with $b_{01} \neq 0$.

Renaming $c := a_{01}b_{10}$ if $b_{10} \neq 0$, we can write

$$\begin{aligned}\alpha(x) &= \Gamma_1(x, y) = \frac{1 + c}{b_{01}}x + \frac{c}{b_{10}}y, \\ \alpha(y) &= \Gamma_2(x, y) = b_{10}x + b_{01}y.\end{aligned}$$

For the particular polynomials $P(x, y) = x, Q(x, y) = y = R(x, y)$, the coefficient in t of $\mathfrak{as}_{\star_\alpha, \alpha}(P, Q, R)$ vanishes only if $c = 0$ and $b_{01} = 0$ or if $c = 0$ and $b_{10} = 0$, which is not possible.

If $b_{10} = 0$ then the morphism α satisfies

$$\begin{aligned}\alpha(x) &= \Gamma_1(x, y) = \frac{1}{b_{01}}x + a_{01}y, \\ \alpha(y) &= \Gamma_2(x, y) = b_{01}y,\end{aligned}$$

and the coefficient in t of $\mathfrak{as}_{\star_\alpha, \alpha}(P, Q, R)$ vanishes only if $a_{01} = 0$. In that case, the morphism α is as in the Proposition 4.9, and thus gives a deformation of the Moyal-Weyl star-product.

Finally, the only morphisms α of degree 1 which give raise to a deformation of the Moyal-Weyl star-product are as in the Proposition 4.9.

4.5.2 Degree 2 case

In degree 2, computations done with the computer algebra system *Mathematica* also led to the case given by the Proposition 4.9.

We conjecture that the only morphisms α of the Proposition 4.11 are the one found in the Proposition 4.9.

An interesting question would be to know if twisting by a morphism α is functorial. If this is the case, the previous conjecture would be true and we would have the following commutative diagram

$$(4.18) \quad \begin{array}{ccc} (A, \mu, \{ , \}) & \xrightarrow{\text{twist}_\alpha} & (A, \mu_\alpha, \{ , \}_\alpha, \alpha) \\ \text{Kont} \updownarrow & & \updownarrow \text{Kont} \\ (A, \star) & \xrightarrow{\text{twist}_\alpha} & (A, \star_\alpha, \alpha) \end{array}$$

with Kont the Kontsevich bijection between the set of equivalence classes of Poisson brackets on the commutative $\mathbb{K}[[t]]$ -algebra $A[[t]]$ and the set of equivalence classes of star products, and $\text{twist}_\alpha : \mathbf{Poiss} \rightarrow \mathbf{Hom-Poiss}$ the functor from the category \mathbf{Poiss} of Poisson algebras to the category $\mathbf{Hom-Poiss}$ of Hom-Poisson algebras. More generally, twist_α is a functor from \mathbf{Struct} to $\mathbf{Hom-Struct}$, where \mathbf{Struct} is a category of structures such as associative algebras \mathbf{Ass} , Lie algebras \mathbf{Lie} , Poisson algebra \mathbf{Poiss} , and so on, and $\mathbf{Hom-Struct}$ the corresponding category of Hom-structures.

References

- [1] N. AIZAWA AND H. SATO, q -deformation of the Virasoro algebra with central extension, *Physics Letters B*, *Phys. Lett. B* **256**, no. 1, 185–190 (1991).
- [2] F. AMMAR, Z. EJBEHI AND A. MAKHLOUF, *Cohomology and Deformations of Hom-algebras*, *Journal of Lie Theory* **21** No. 4, (2011) 813–836 .
- [3] F. AMMAR AND A. MAKHLOUF, *Hom-Lie algebras and Hom-Lie admissible superalgebras*, *J. Algebra*, Vol. **324** (7), (2010) 1513–1528.
- [4] F. BAYEN, M. FLATO, C. FRONSDAL, A. LICHNEROWICZ AND D. STERNHEIMER, *Deformation Theory and Quantization. Deformations of Symplectic Structures*, *Annals of Physics*, **111** (1978) 61–110.
- [5] M. BORDEMANN, The deformation quantization of certain super-Poisson brackets and BRST cohomology. *Conference Moshe Flato 1999*, Vol. II (Dijon), 45-68, *Math. Phys. Stud.*, **22**, Kluwer Acad. Publ., Dordrecht, 2000.

- [6] ——— *Deformation quantization: a mini-lecture. Geometric and topological methods for quantum field theory*, 3–38, Contemp. Math., **434**, Amer. Math. Soc., Providence, RI, 2007.
- [7] M. BORDEMANN, A. MAKHLOUF AND T. PETIT, *Déformation par quantification et rigidité des algèbres enveloppantes*. J. algebra **285** , no. 2 (2005), 623–648.
- [8] S. CAENEPEEL AND I. GOYVAERTS , *Monoidal Hom-Hopf algebras*, Comm. Alg. **39** (2011) 2216–2240.
- [9] M. CHAICHIAN, P. KULISH AND J. LUKIERSKI, *q-Deformed Jacobi identity, q-oscillators and q-deformed infinite-dimensional algebras*, Phys. Lett. B **237** , no. 3-4, (1990) 401–406.
- [10] V. CHARI AND A. PRESSLEY, *A guide to quantum groups*, 667 pp, Cambridge University Press (1994).
- [11] T. L. CURTRIGHT AND C. K. ZACHOS, *Deforming maps for quantum algebras*, Phys. Lett. B **243**, no. 3, 237–244 (1990).
- [12] K. DEKKAR AND A. MAKHLOUF, *Cohomology and Deformations of Hom-Bialgebras and Hom-Hopf algebras*, In preparation.
- [13] V. G. DRINFEL'D, *Hopf algebras and the quantum Yang-Baxter equation*, Soviet Math. Doklady, **32**, 254–258 (1985).
- [14] M. GERSTENHABER, *On the deformation of rings and algebras*, Ann of Math. **79** (1) (1964), 59–108.
- [15] M. GERSTENHABER AND S. D. SCHACK *algebras, bialgebras, quantum groups, and algebraic deformations*. Deformation theory and quantum groups with applications to mathematical physics (Amherst, MA, 1990), 51–92, Contemp. Math., 134, Amer. Math. Soc., Providence, RI, 1992.
- [16] Fialowski A.: *Deformation of Lie algebras*, Math USSR Sbornik, vol. **55**, 467–473 (1986).
- [17] Y. FREGIER, A. GOHR AND S.D. SILVESTROV, *Unital algebras of Hom-associative type and surjective or injective twistings*, J. Gen. Lie Theory Appl. Vol. **3** (4), (2009) 285–295.
- [18] A. GOHR, *On Hom-algebras with surjective twisting*, J. Algebra **324** (2010) 1483–1491.
- [19] M. GOZE AND E. REMM, *Poisson algebras in terms of non-associative algebras*, J. Algebra **320** (2008) 294–317.

- [20] J. T. HARTWIG, D. LARSSON AND S.D. SILVESTROV, Deformations of Lie algebras using σ -derivations, *J. Algebra* **295**, 314–361 (2006).
- [21] N. HU, q -Witt algebras, q -Lie algebras, q -holomorph structure and representations, *Algebra Colloq.* **6**, no. 1, 51–70 (1999).
- [22] C. KASSEL, Cyclic homology of differential operators, the Virasoro algebra and a q -analogue, *Commun. Math. Phys.* **146** (1992), 343–351.
- [23] ——— Quantum groups, *Graduate Texts in Mathematics*, 155. Springer-Verlag, New York (1995).
- [24] M. KONTSEVICH, *Deformation Quantization of Poisson Manifolds I*, *Letters in Mathematical Physics* Volume 66, Number 3, (2003)157–216.
- [25] D. LARSSON AND S.D. SILVESTROV, Quasi-hom-Lie algebras, Central Extensions and 2-cocycle-like identities, *J. Algebra* **288**, 321–344 (2005).
- [26] A. MAKHLOUF, Degeneration, rigidity and irreducible components of Hopf algebras, *Algebra Colloquium*, vol **12** (2), 241–254 (2005).
- [27] ——— Algèbre de Hopf et renormalisation en théorie quantique des champs, In "Théorie quantique des champs : Méthodes et Applications", *Travaux en Cours*, Hermann Paris, 191–242 (2007).
- [28] ——— *Paradigm of Nonassociative Hom-algebras and Hom-superalgebras*, *Proceedings of Jordan Structures in Algebra and Analysis Meeting*, Eds: J. Carmona Tapia, A. Morales Campoy, A. M. Peralta Pereira, M. I. Ramirez Alvarez, Publishing house: Circulo Rojo (2010), 145–177.
- [29] A. MAKHLOUF AND S. SILVESTROV, *Hom-algebra structures*, *Journal of Generalized Lie Theory and Applications*, Volume 2, No. 2 (2008), 51–64.
- [30] ——— *Hom-Lie admissible Hom-coalgebras and Hom-Hopf algebras*, Published as Chapter 17, pp 189–206, S. Silvestrov, E. Paal, V. Abramov, A. Stolin, (Eds.), *Generalized Lie theory in Mathematics, Physics and Beyond*, Springer-Verlag, Berlin, Heidelberg, (2008).
- [31] ——— *Notes on Formal deformations of Hom-Associative and Hom-Lie algebras*, *Forum Mathematicum*, vol. **22** (4) (2010) 715–759.
- [32] ——— *Hom-Algebras and Hom-Coalgebras*, *J. of Algebra and its Applications*, Vol. **9**, (2010).
- [33] M. MARKL AND E. REMM, *Algebras with one operation including Poisson and other Lie-admissible algebras*, *Journal of Algebra* **299**, Issue 1, (2006) 171–189

- [34] A. NIJENHUIS AND W. RICHARDSON, *Cohomology and deformations in graded Lie Algebras*, Bulletin of the American Mathematical Society, **72**, (1966) 1–29
- [35] S.Q. OH AND H.M. PARK, *Duality of co-Poisson Hopf algebras*, Bulletin of the Korean Mathematical Society, Volume 48 No. 1 (2011) 17–21
- [36] G. ORTENZI, V. RUBTSOV AND S.R. TAGNE PELAP, *On the Heisenberg invariance and the Elliptic Poisson tensors*, arXiv:1001.4422v2 (2010).
- [37] Y. SHENG, *Representations of hom-Lie algebras*, Algebra and Representation Theory, DOI:10.1007/s10468-011-9280-8 (2011).
- [38] M.E. SWEEDLER, *Hopf algebras*, W.A. Benjamin, Inc. Publishers, New York, 1969.
- [39] D. YAU, *Enveloping algebra of Hom-Lie algebras*, J. Gen. Lie Theory Appl. **2** (2008) 95–108.
- [40] ——— *Hom-algebras and homology*, J. Lie Theory **19** (2009) 409–421.
- [41] ——— *The Hom-Yang-Baxter equation, Hom-Lie algebras, and quasi-triangular bialgebras*, J. Phys. A **42** (2009) 165–202.
- [42] ——— *Hom-bialgebras and comodule Hom-algebras*, Int. E. J. Alg. **8** (2010) 45–64.
- [43] ——— *The Hom-Yang-Baxter equation and Hom-Lie algebras*, J. Math. Phys. **52** (2011) 053502.
- [44] ——— *The classical Hom-Yang-Baxter equation and Hom-Lie bialgebras*, e-Print arXiv:0905.1890 (2009).
- [45] ——— *Non-commutative Hom-Poisson algebras*, e-Print arXiv:1010.3408 (2010).

Martin Bordemann, Olivier Elchinger and Abdenacer Makhoulouf
Université de Haute Alsace
Laboratoire de Mathématiques, Informatique et Applications
4, rue des Frères Lumière, 68093 Mulhouse cedex France
Martin.Bordemann@uha.fr
Olivier.Elchinger@uha.fr
Abdenacer.Makhoulouf@uha.fr

