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On the integration of $\mathcal{LA}$-groupoids and duality for Poisson groupoids [1] [2]

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Abstract

In this note a functorial approach to the integration problem of an $\mathcal{LA}$-groupoid to a double Lie groupoid is discussed. To do that, we study the notions of fibered products in the categories of Lie groupoids and Lie algebroids, giving necessary and sufficient conditions for the existence of such. In particular, it turns out, that the fibered product of Lie algebroids along integrable morphisms is always integrable by a fibered product of Lie groupoids. We show that to every $\mathcal{LA}$-groupoid with integrable top structure one can associate a differentiable graph in the category of Lie groupoids, which is an integrating double Lie groupoid, whenever some lifting conditions for suitable Lie algebroid homotopies are fulfilled; the result specializes to the case of a Poisson groupoid, yielding a symplectic double groupoid, provided our conditions on the associated $\mathcal{LA}$-groupoid are satisfied.

Introduction

In recent years, fundamental questions in Lie theory for Lie algebroids and Lie groupoids have been answered; namely, optimal generalizations of Lie's theorems have been discovered. Examples of non-integrable Lie algebroids already appeared in [AM] and the problem to find general integrability conditions has been standing for a long time.

The −quite non-trivial− theory of morphisms of Lie algebroids was developed by Higgins and Mackenzie in [HM]. Later on Mackenzie and Xu proved [MX2] that morphisms of integrable Lie algebroids are integrable to morphisms of Lie groupoids, provided the domain groupoid has 1-connected source fibres. An independent proof by Moerdijk and Mrčun appeared in [MM]. In the same paper the authors also show that to every source connected Lie groupoid one can associate a unique source 1-connected "cover" with the same Lie algebroid; moreover they prove that every Lie subalgebroid of an integrable Lie algebroid A is integrable by an (only)

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immersed subgroupoid of the source 1-connected integration of A. The source 1-connected cover of a source connected Lie groupoid $\mathcal{G}$ is obtained as the quotient of the monodromy groupoid $\mathsf{Mon}(\mathcal{G}, \mathrm{s})$ associated with the source foliation on $\mathcal{G}$ with respect to the natural action by right translation of $\mathcal{G}$ itself. It turns out [CF1], that the Lie groupoid $\mathsf{Mon}(\mathcal{G}, \mathrm{s})/\mathcal{G}$ can be equivalently described as the quotient of the so called $\mathcal{G}$-paths, paths along the source fibers starting from the base manifold, by homotopy within the source fibers, relative to the end points. Crainic and Fernandes showed that both the notions of $\mathcal{G}$-paths and their homotopy can be characterized in terms of the Lie algebroid A of $\mathcal{G}$; namely, $\mathcal{G}$-paths are in bijective correspondence with A-paths, i.e. morphisms of Lie algebroids $TI \to A$, and $\mathcal{G}$-paths are homotopic iff the corresponding A-paths are A-homotopic, being A-homotopopies morphisms of Lie algebroids $TI^{\times 2} \to A$, satisfying suitable boundary conditions. The quotient $\mathcal{W}(A) := \{A\text{-}paths\}/A\text{-}homotopy$, a.k.a. the *Weinstein groupoid*, carries a natural groupoid structure, induced, roughly, by concatenation of paths; it is always a topological groupoid and Crainic and Fernandes finally delivered a necessary and sufficient integrability condition for Lie algebroids, which is to be understood as the obstruction to put a smooth structure on the associated Weinstein groupoid.
The construction of the Weinstein groupoid was anticipated by Cattaneo and Felder [CF] in the special case of the Lie algebroid of a Poisson manifold. Their approach involves the symplectic reduction of the phase space of the Poisson sigma model and yields the symplectic groupoid of the target Poisson manifold, in the integrable case.

In this paper we study (part of) the categorified version of this story. Ehresmann's categorification of a groupoid is a groupoid object in the category of groupoids; this is a symmetric notion and it makes sense to regard such a structure as a "double groupoid". A double Lie groupoid is, essentially, a "Lie groupoid in the category of Lie groupoids"; one can apply the Lie functor to the object of a double Lie groupoid, to obtain an $\mathcal{LA}$-groupoid, i.e. a "Lie groupoid in the category of Lie algebroids". The application of the Lie functor can still be iterated; the result, a double Lie algebroid, is the best approximation to what one would mean as a "Lie algebroid in the category of Lie algebroids". Such double structures do arise in nature, especially from Poisson geometry and the theory of Poisson actions.

After reviewing the main definitions and known integrability results related to Lie bialgebroids and Poisson groupoids (§1), we address the integration problem of an $\mathcal{LA}$-groupoid to a double Lie groupoid. In §2 we study the fibred products of Lie algebroids and Lie groupoids. We show that, whenever a fibered product of Lie algebroids exists as a vector bundle, it carries a unique natural Lie algebroid structure; the analogous property does not hold for Lie groupoids. We find, however a necessary and sufficient transversality condition for fibered products of Lie groupoids to stay in the category. We develop our integration approach in §3, also considering the case of the $\mathcal{LA}$-groupoid of a Poisson groupoid in relation

with duality issues (§4). We introduce differentiable graphs with structure, Lie groupoids "without a multiplication", and show that an $\mathcal{LA}$-groupoid with integrable top Lie algebroid is always integrable to an invertible unital graph in the category of Lie groupoids; moreover, we prove a natural integrability result for fibered products of integrable Lie algebroids. As a consequence the integrating graph of an $\mathcal{LA}$-groupoid can be endowed with a further compatible multiplication making it a double Lie groupoid, under some connectivity assumptions on its second and third nerve. Surprisingly, the connectivity assumptions are implied by some suitable lifting conditions for Lie algebroid paths and Lie algebroid homotopies, depending only on the original $\mathcal{LA}$-groupoid. We shall remark, however, that our requirements are far from being necessary integrability conditions and appear quite restrictive.
Lastly, we comment on an alternative approach to the -2 steps in $1-$ integration of a Lie bialgebroid to a symplectic double groupoid, within the framework of symplectic reduction of the Courant sigma model.

Notations and conventions

We denote with s and t the source and target maps of a Lie groupoid, with ε the unit section, with ι the inversion and with μ the partial multiplication. The anchor of a Lie algebroid is typically denoted with ρ. Nowhere in this paper $\overline{P}$ denotes the opposite Poisson structure on a Poisson manifold P. "Fibered product" is meant as a categorical pullback; with pullback it is meant "pullback along a map", such as the vector bundle pullback. We shall denote with $\Gamma(f)$ the graph of a map $f : M \to N$ and regard it as a fibered product $M\,{}_f\!\times N \equiv M\,{}_f\!\times_{\mathsf{id}_N} N$. If two smooth maps $f_{1,2} : M^{1,2} \to N$ are transversal, we shall write $f_1 \pitchfork f_2$. For two vector bundle maps $\phi_{1,2} : E^{1,2} \to F$ to be transversal, $\phi_1 \pitchfork \phi_2$, means that ϕ_1 and ϕ_2 are transversal as smooth maps, so are the corresponding base maps $f_{1,2} : M_{1,2} \to N$ and $\phi_1 - \phi_2 : (E^1 \times E^2)|_{M^1\,{}_{f_1}\!\times_{f_2} M^2} \to F$ has maximal rank, so that the fibered product $E^1\,{}_{\phi_1}\!\times_{\phi_2} E^2$ carries a vector bundle structure over the fibered product $M^1\,{}_{f_1}\!\times_{f_2} M^2$.

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1 $\mathcal{LA}$-groupoids, (symplectic) double Lie groupoids and Poisson groupoids

A double Lie groupoid is a Lie groupoid object in the category of Lie groupoids.

Definition 1.1 ([M1]). A double Lie groupoid $\mathsf{D} := (\mathcal{D}, \mathcal{H}, \mathcal{V}, M)$

$$\begin{tikzcd} \mathcal{D} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{V} \arrow[d, shift left] \arrow[d, shift right] \\ \mathcal{H} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

is a groupoid object in the category of groupoids (i.e. a double groupoid in the sense of Ehresmann), such that $\mathcal{D} \rightrightarrows \mathcal{V}$, $\mathcal{D} \rightrightarrows \mathcal{H}$, $\mathcal{H} \rightrightarrows M$, $\mathcal{V} \rightrightarrows M$ are Lie groupoids and the double source map

$$\mathbb{S} \doteq (\mathrm{s}_H, \mathrm{s}_V) : \mathcal{D} \to \mathcal{H}\ {}_{\mathrm{s}_h}\!\times_{\mathrm{s}_v} \mathcal{V}$$

is submersive[4].

The definition is symmetric and the total space of a double Lie groupoid can be regarded as a groupoid object either horizontally or vertically; the groupoid vertical, resp. horizontal, structural maps (unit section, source, target, inversion and multiplication) are morphisms of Lie groupoids for the vertical, resp. horizontal, structures. Note that the submersivity condition on the double source map makes the domains of the top multiplications Lie groupoids (see proposition (2.1) for a justification of this fact).

Applying the Lie functor horizontally, or vertically, yields an $\mathcal{LA}$-groupoid.

Definition 1.2 ([M1]). An $\mathcal{LA}$-groupoid $\Omega := (\Omega, A, \mathcal{G}, M)$

$$\begin{tikzcd} \Omega \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\ A \arrow[r] & M \end{tikzcd}$$

is a groupoid object in the category of Lie algebroids, such that $\Omega \rightrightarrows A$ and $\mathcal{G} \rightrightarrows M$ are Lie groupoids and the double source map

$$\$ \doteq (\hat{\mathrm{s}}, \mathrm{Pr}) : \Omega \to A\ {}_{\mathrm{pr}}\!\times_{\mathrm{s}} \mathcal{G}$$

is a surjective submersion.

[4]In [M1] the double source map is required to be also surjective; this condition does not really play a role in the study of the internal structure of a double Lie groupoid and the descent to double Lie algebroids. Moreover, there are interesting examples, such as Lu and Weinstein's double of a Poisson group (1.10) for instance, which do not fulfill the double source surjectivity condition.

An $\mathcal{LA}$-groupoid is a double groupoid for the horizontal additive groupoids and the double source map should be understood with respect to this structure. As a direct consequence from the definition, the vector bundle projection, zero section, fibrewise addition and scalar multiplication of $\Omega \to \mathcal{G}$ are morphisms of Lie groupoids over the corresponding maps of $A \to M$. Note that there is no natural way of characterizing the Lie algebroid bracket of Ω as a morphism of Lie algebroids over the bracket of A. On the other hand, an $\mathcal{LA}$-groupoid is indeed a Lie groupoid in the category of Lie algebroids.

Example 1.3. The typical examples are: for any Lie groupoid $\mathcal{G} \rightrightarrows M$

$$\begin{tikzcd} \mathcal{G}\times\mathcal{G} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\ M\times M \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd} \qquad \begin{tikzcd} T\mathcal{G} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\ TM \arrow[r] & M \end{tikzcd},$$

where the top vertical groupoid is a direct product and the horizontal groupoids are pair groupoids, in the first case, while in the second, the top groupoid is the tangent prolongation (each structural map is the tangent of the corresponding map of $\mathcal{G}$).

Applying the Lie functor once more yields a double Lie algebroid (See [M1, M7] for a definition). The notions of morphisms and sub-objects of double structures are the obvious ones.

Lie theory "from double Lie groupoids to double Lie algebroids" has been developed to a satisfactory extent in recent years by Mackenzie [M1]-[M5], [M7]. Integrability results for double Lie algebroids to $\mathcal{LA}$-groupoids and for $\mathcal{LA}$-groupoids to double Lie groupoids are known only for a restricted class of examples arising from Poisson geometry, the main ones we are about to sketch.

Definition 1.4 ([W])**.** A Poisson groupoid is a Lie Groupoid $\mathcal{P} \rightrightarrows M$ endowed with a compatible Poisson structure $\Pi \in \mathfrak{X}^2(\mathcal{P})$, i.e. such that the graph $\Gamma(\mu) \subset \mathcal{P}^{\times 3}$ of the groupoid multiplication μ is coisotropic with respect to the Poisson structure $\Pi \times \Pi \times -\Pi$.

A Poisson groupoid with a non-degenerate Poisson structure is a symplectic groupoid in the usual sense (i.e. $\Gamma(\mu)$ is Lagrangian, by counting dimensions). For any Poisson groupoid $(\mathcal{P}, \Pi) \rightrightarrows M$ [W]:

(i) The unit section $\varepsilon : M \hookrightarrow \mathcal{P}$ is a closed coisotropic embedding;

(ii) The inversion map $\iota : \mathcal{P} \to \mathcal{P}$ is an anti-Poisson diffeomorphism;

(iii) The source invariant functions and the target invariant functions define commuting anti-isomorphic Poisson subalgebras of $\mathcal{C}^\infty(\mathcal{P})$.

As an easy consequence of property (iii) above, the base manifold of a Poisson groupoid carries a unique Poisson structure making the source map Poisson and the target map anti-Poisson. A symplectic groupoid provides a *symplectic realization* of the Poisson structure induced on the base (see, for example, [CDW, CF2] for an account on symplectic realizations and symplectic groupoids).

It turns out [MX1], that a Lie groupoid $\mathcal{P} \rightrightarrows M$ with a Poisson structure Π is a Poisson groupoid iff

$$\begin{tikzcd} T^*\mathcal{P} \arrow[r, "\Pi^\sharp"] \arrow[d, shift left] \arrow[d, shift right] & T\mathcal{P} \arrow[d, shift left] \arrow[d, shift right] \\ A^* \arrow[r] & TM \end{tikzcd} \quad \text{is a morphism of groupoids.} \tag{1.1}$$

The base map in the above diagram is the restriction of the Poisson anchor to $N^*M = \mathsf{Ann}_{T^*\mathcal{P}}TM$, which is to be canonically identified with the dual bundle to the Lie algebroid A of $\mathcal{P}$. The cotangent prolongation groupoid [CDW] $T^*\mathcal{G} \rightrightarrows A^*$, can be defined for any Lie groupoid $\mathcal{G} \rightrightarrows M$ and it is the symplectic groupoid of the fibrewise linear Poisson structure induced from the Lie algebroid A of $\mathcal{G}$ on A^*. If $\mathcal{P}$ is a Poisson groupoid A^*, as the conormal bundle to a coisotropic submanifold, carries a Lie algebroid structure over M and

$$\begin{tikzcd} T^*\mathcal{P} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{P} \arrow[d, shift left] \arrow[d, shift right] \\ A^* \arrow[r] & M \end{tikzcd} \quad \text{is an } \mathcal{LA}\text{-groupoid.} \tag{1.2}$$

In fact (1.1) is the compatibility condition between horizontal anchors and vertical Lie groupoids. The compatibility with the Lie algebroid brackets can be shown as a consequence of the duality between $\mathcal{PVB}$-groupoids and $\mathcal{LA}$-groupoids (see [M2] for details).

Recall that, if $(\mathcal{P}, \Pi) \rightrightarrows M$ is a Poisson groupoid, (A, A^*) is a Lie bialgebroid [MX1]; that is, the Lie algebroid structures on A and A^* are compatible, in the sense that [K] $(\Gamma(\wedge^\bullet A^*), \wedge, \mathrm{d}_A, [\,,\,]_{A^*})$ is a differential Gerstenhaber algebra for the Lie algebroid differential d_A induced by A and the graded Lie bracket $[\,,\,]_{A^*}$ on $\Gamma(\wedge^\bullet A^*)[1]$ induced by A^*. The notion of Lie bialgebroid is self dual ((A, A^*) is a Lie bialgebroid iff so is (A^*, A)) and the flip $(A^*, \bar{A})$ of a Lie bialgebroid (invert signs of the anchor and bracket of A) is also a Lie bialgebroid. This leads to a notion of duality for Poisson groupoids, essentially introduced in [W].

Definition 1.5 ([M2]). Poisson groupoids $(\mathcal{P}_\pm, \Pi_\pm) \rightrightarrows M$ are in weak duality if the Lie bialgebroid of $\mathcal{P}_+$ is isomorphic to the flip of the Lie bialgebroid of $\mathcal{P}_-$.

There is an important integrability result for Lie bialgebroids.

Theorem 1.6 ([MX2]). *For any Lie bialgebroid $(A, A^*) \to M$, with A integrable, there exists a unique Poisson structure $\Pi \in \mathfrak{X}^2(\mathcal{P})$ on the source 1-connected Lie groupoid $\mathcal{P} \rightrightarrows M$ of A, such that*

1. *$(\mathcal{P}, \Pi) \rightrightarrows M$ is a Poisson groupoid,*
2. *The Lie algebroid on A^* coincides with that induced by Π.*

Last result, in view of [M2, M3, M7], can be interpreted as an integrability result for the cotangent double (Lie algebroid) of a Lie bialgebroid

$$\begin{tikzcd} T^*A^* \arrow[r] \arrow[d] & A \arrow[d] \\ A^* \arrow[r] & M \end{tikzcd}$$

to the $\mathcal{LA}$-groupoid (1.2) of the corresponding source 1-connected Poisson groupoid. See also [M6] for another example of an integrable double Lie algebroid arising from the Poisson action of a Poisson group.
An easy consequence of theorem (1.6) is the following.

Lemma 1.7. *Every integrable Poisson groupoid $(\mathcal{P}, \Pi) \rightrightarrows M$ has a unique source 1-connected weak dual Poisson groupoid $(\overline{\mathcal{P}}, \overline{\Pi}) \rightrightarrows M$.*

Proof. Recall from [MM] that a Lie subalgebroid of an integrable Lie algebroid is integrable. Since $\mathcal{P}$ is an integrable Poisson manifold $T^*\mathcal{P}$ is an integrable Lie algebroid and so is A^* (embed A^* in $T^*\mathcal{P}$ using the unit section of the cotangent prolongation groupoid). Apply theorem (1.6) to the flip of the Lie bialgebroid of $\mathcal{P}$. □

A stronger notion of duality for Poisson groupoids arises from double groupoids.

Definition 1.8 ([M2])**.** A symplectic double groupoid is a double Lie groupoid, whose total space is endowed with a symplectic form, which is compatible with the top horizontal and vertical Lie groupoids.

It follows [M2], that the Poisson structures on the total spaces of the side groupoids induced by the top horizontal and vertical groupoids are in weak duality. This motivates the following definition.

Definition 1.9.[5] Two Poisson groupoids $(\mathcal{P}_\pm, \Pi_\pm) \rightrightarrows M$ are in strong duality if there exists a symplectic double groupoid

$$\begin{tikzcd} \mathcal{S} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{P}_+ \arrow[d, shift left] \arrow[d, shift right] \\ \mathcal{P}_- \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

with the given groupoids as side structures, such that the symplectic form induces the given Poisson structures. The double of an integrable Poisson groupoid is a symplectic double groupoid $\mathcal{S}$ realizing a strong duality of $\mathcal{P}$ and its weak dual Poisson groupoid $\overline{\mathcal{P}}$.

[5]This notion was suggested to the author by K. Mackenzie in a private discussion (2005).

Thus, strongly dual Poisson groupoids admit a simultaneous integration (and symplectic realization); the natural questions to answer are:

· *Do Poisson groupoids integrate to symplectic double groupoids*?

And, more generally,

· *Does weak duality imply strong duality for integrable Poisson groupoids*?

A positive answer to the first question was given in [LW] by Lu and Weinstein in the case of integrable Poisson groups, and [LP] by Li and Parmentier in the case of a class of coboundary dynamical Poisson groupoids.

Example 1.10. [LW] Let $(\mathfrak{g}, \mathfrak{g}^*)$ be the tangent Lie bialgebra of a 1-connected Poisson group G and $\overline{G}$ be the weak dual. The sum $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ carries a natural Lie algebra structure, obtained by a double twist of the brackets on $\mathfrak{g}$ and $\mathfrak{g}^*$, the *Drinfel'd double* of $(\mathfrak{g}, \mathfrak{g}^*)$; let D be the 1-connected integration of such. Denote with $\lambda : G \hookrightarrow D$, resp. $\rho : \overline{G} \hookrightarrow D$ the integrations of $\mathfrak{g} \hookrightarrow \mathfrak{d}$, resp. $\mathfrak{g}^* \hookrightarrow \mathfrak{d}$. One can show that D has a compatible Poisson structure π_D, which happens to be non-degenerate on the submanifold of elements d, admitting a decomposition $d = \lambda(g_+)\rho(\overline{g}_+) = \rho(\overline{g}_-)\lambda(g_-)$, $g_\pm \in G$ and $\overline{g}_\pm \in \overline{G}$. Moreover there is also a natural double Lie groupoid

$$\begin{array}{ccc} \mathcal{D} & \rightrightarrows & G \\ \downdownarrows & & \downdownarrows \\ \overline{G} & \rightrightarrows & \bullet \end{array} \qquad\qquad \mathcal{D} = G \times \overline{G} \times \overline{G} \times G \quad .$$

It turns out that the double subgroupoid, whose total space is $\mathcal{S} = \{(g_+, \overline{g}_+, \overline{g}_-, g_-)| \lambda(g_+)\rho(\overline{g}_+) = \rho(\overline{g}_-)\lambda(g_-)\}$ carries a compatible symplectic form, inducing the Poisson structures on G and $\overline{G}$, which is the inverse of the pullback of π_D, under the natural local diffeomorphism $\mathcal{S} \to D$. Note that $\mathcal{S}$ is, in general, neither vertically, nor horizontally source (1-)connected.

2 Fibered products in the categories of Lie algebroids and Lie groupoids

Recall from [HM] that, given Lie algebroids $A^{1,2} \to M^{1,2}$, with anchors $\rho_{1,2}$ and brackets $[\cdot,\cdot]_{1,2}$ a morphism of Lie algebroids is a smooth vector bundle map $\varphi : A^1 \to A^2$ over a base map $f : M^1 \to M^2$, satisfying the natural anchor compatibility condition, $\rho_2 \circ \phi = \mathrm{d}f \circ \rho_1$, and a bracket compatibility condition. The bracket compatibility can be expressed in a few equivalent ways, using decompositions of sections or connections: pick a Koszul connection ∇ for $A^2 \to M^2$, denote with $f^+\nabla$ the induced connection on the pullback bundle $f^+A^2 \to M^1$ and with $\varphi^!$ the bundle map $A^1 \to f^+A^2$ induced by φ, the condition is

$$\varphi^![a,b]_1 = f^+\nabla_{\rho_1(a)}\varphi^!b - f^+\nabla_{\rho_1(b)}\varphi^!a - f^+\tau^\nabla(\varphi^!a, \varphi^!b) \quad , \quad a, b \in \Gamma(A^1) \qquad ,$$

where $f^+\tau^\nabla$ is the pullback of the torsion tensor of ∇. With the above definition one can show that there is a category of Lie algebroids, with direct products and pullbacks (under natural transversality conditions). More invariantly, a Lie algebroid structure on a vector bundle $A \to M$ is equivalent to a differential on the graded algebra $\mathsf{C}^\bullet(A) := (\Gamma(\wedge^\bullet A^*), \wedge)$ [V]; a vector bundle map $\phi : A^1 \to A^2$ is then a morphism of Lie algebroids iff [K] the induced map $\mathsf{C}^\bullet(A^2) \to \mathsf{C}^\bullet(A^1)$ is a chain map.
A Lie subalgebroid B of A is a vector subbundle such that the inclusion $B \hookrightarrow A$ is a morphism of Lie algebroids.
The Lie algebroid of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is the vector bundle $T^{\mathrm{s}}_M\mathcal{G}$ (the restriction of the kernel of the tangent source map to the base manifold) endowed with a bracket induced from that of right invariant sections of $T^{\mathrm{s}}\mathcal{G}$. Similarly restricting the tangent map of a morphism of Lie groupoids $\mathcal{G}^1 \to \mathcal{G}^2$ to $T^{\mathrm{s}^1}_{M^1}\mathcal{G}^1 \to T^{\mathrm{s}^2}_{M^2}\mathcal{G}^2$, one obtains a morphism of Lie algebroids. That is, there exists a Lie functor from the category of Lie groupoids to that of Lie algebroids.
Moreover the Lie functor preserves direct products and pullbacks; note, however, that fibered products of Lie groupoids do not always exist.

Proposition 2.1. [6] *Consider morphisms of Lie groupoids $\varphi_{1,2} : \mathcal{G}^{1,2} \to \mathcal{H}$ over $f_{1,2} : M^{1,2} \to N$, such that $\varphi_1 \pitchfork \varphi_2$ and $f_1 \pitchfork f_2$. Then, the manifold fibered product $\mathcal{G}^1 {}_{\varphi_1}\!\times_{\varphi_2} \mathcal{G}^2$ exists and the natural (smooth) groupoid structure over $M^1 {}_{f_1}\!\times_{f_2} M^2$ induced from the direct product $\mathcal{G}^1 \times \mathcal{G}^2$ is that of a* Lie *groupoid iff the* source transversality condition

$$\mathrm{d}\varphi_1 T^{\mathrm{s}_1}_{g_1}\mathcal{G}^1 + \mathrm{d}\varphi_2 T^{\mathrm{s}_2}_{g_2}\mathcal{G}^2 = T^{\mathrm{s}}_h\mathcal{H} \quad , \quad \varphi_1(g_1) = h = \varphi_2(g_2) \tag{2.1}$$

is satisfied for all $(g_1, g_2) \in \mathcal{G}^1 {}_{\varphi_1}\!\times_{\varphi_2} \mathcal{G}^2$. In this case, $\mathcal{G}^1 {}_{\varphi_1}\!\times_{\varphi_2} \mathcal{G}^2$ is a fibered product in the category of Lie groupoids.

Proof. We have to prove that the source map of $\mathcal{G} \equiv \mathcal{G}^1 {}_{\varphi_1}\!\times_{\varphi_2} \mathcal{G}^2 \to M \equiv M^1 {}_{f_1}\!\times_{f_2} M^2$ is submersive iff (2.1) holds; universality is then manifest. Let $q_{1,2} = \mathrm{s}_{1,2}(g_{1,2})$, $q = \mathrm{s}(h)$, for any $(g_1, g_2) \in \mathcal{G}$, $h = \varphi_{1,2}(g_{1,2})$, applying the snake lemma to the exact commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & T^{\mathrm{s}_1}_{g_1}\mathcal{G}^1 \oplus T^{\mathrm{s}_2}_{g_2}\mathcal{G}^2 \arrow[r] \arrow[d, "\Phi_{12}"] & T_{g_1}\mathcal{G}^1 \oplus T_{g_2}\mathcal{G}^2 \arrow[r, "\mathrm{ds}_1 \times \mathrm{ds}_2"] \arrow[d, "\mathrm{d}\varphi_1 - \mathrm{d}\varphi_2"] & T_{q_1}M^1 \oplus T_{q_2}M^2 \arrow[r] \arrow[d, "\mathrm{d}f_1 - \mathrm{d}f_2"] & 0 \\
0 \arrow[r] & T^{\mathrm{s}}_h\mathcal{H} \arrow[r] & T_h\mathcal{H} \arrow[r, "\mathrm{ds}"'] & T_qN \arrow[r] & 0
\end{tikzcd}$$

yields a connecting arrow ∂ and a long exact sequence

$$0 \longrightarrow \ker \Phi_{12} \longrightarrow T_{(g_1,g_2)}\mathcal{G} \xrightarrow{\mathrm{ds}_{12}} T_{(q_1,q_2)}M \xrightarrow{\partial} \operatorname{coker} \Phi_{12} \longrightarrow 0 \quad ,$$

where $\Phi_{12} := (\mathrm{d}\varphi_1 - \mathrm{d}\varphi_2)|_{T^{\mathrm{s}_1}_{g_1}\mathcal{G}^1 \oplus T^{\mathrm{s}_2}_{g_2}\mathcal{G}^2}$ and $\mathrm{s}_{12} = (\mathrm{s}_1 \times \mathrm{s}_2)|_{\mathcal{G}}$. The result follows. $\square$

[6]Stronger sufficient conditions for the existence of fibred products of Lie groupoids appeared in [Mb].

Generalizing slightly the constructions of [HM] allows to introduce abstract fibered products of Lie algebroids[7]. Recall from [HM] that the *pullback algebroid*, $f^{++}A \to N$, along a smooth map $f : N \to M$ of a Lie algebroid $A \to M$, with anchor ρ, is defined whenever $\mathrm{d}f - \rho^! : TN \oplus f^+A \to TM$ has constant rank (in particular, when f is submersive); the total space of $f^{++}A$ is then $\mathsf{ker}\,(\mathrm{d}f - \rho^!)$, the anchor, denoted with ρ_{++}, being the first projection (see [HM, Mb] for a description of the bracket). Pullback algebroids satisfy the following important universal property.

Proposition 2.2 ([HM])**.** *Let $A^{(\prime)} \to M^{(\prime)}$ be Lie algebroids and $f : N \to M$ a smooth map, such that the pullback $f^{++}A$ exists. For any morphism of Lie algebroids $\phi : A' \to A$ over $g : M' \to M$, such that there is a smooth factorization $g = f \circ h$, for some $h : M' \to N$, there exists a unique morphism $\psi : A' \to f^{++}A$, such that $\phi = f^{++} \circ \psi$, for the natural morphism (a.k.a. the* inductor*) $f^{++} : f^{++}A \to A$.*

Next, consider Lie algebroids A^1, A^2 and B *over the same base* M; given morphisms of Lie algebroids $\phi_{1,2} : A^{1,2} \to B$ over the identity, such that the fibered product $A^1 {}_{\phi_1}\!\times_{\phi_2} A^2$ is a vector bundle, it is possible to introduce the fibered product Lie algebroid (over B in this case), for the vector bundle structure over M. The anchor is $\rho(a_1 \oplus a_2) = \rho_B \circ \phi_1(a_1) = \rho_B \circ \phi_2(a_2)$, for any $a_1 \oplus a_2 \in A_1 {}_{\phi_1}\!\times_{\phi_2} A_2$; the bracket is defined componentwise:

$$[a_1 \oplus a_2, b_1 \oplus b_2] = [a_1, b_1] \oplus [a_2, b_2] \quad , \quad a_1 \oplus a_2, b_1 \oplus b_2 \in \Gamma(M, A_1 {}_{\phi_1}\!\times_{\phi_2} A_2) \quad .$$

Last construction is a straightforward generalization of the *product of Lie algebroids over the same base* in [HM], which is recovered replacing B with TM and $\phi_{1,2}$ with $\rho_{1,2}$.
Given Lie algebroids $A^{1,2} \to M^{1,2}$, denote with M^{12} the direct product $M^1 \times M^2$ and with $\mathrm{pr}_{1,2}$ the projections onto $M^{1,2}$. Since the pullback algebroids $\mathrm{pr}_{1,2}^{++} A^{1,2}$ always exist and the fibered product of manifolds $\mathrm{pr}_1^{++} A^1 {}_{\rho^1_{++}}\!\times_{\rho^2_{++}} \mathrm{pr}_2^{++} A^2$ is to be identified with the vector bundle $A^1 \times A^2 \to M^1 \times M^2$, there is always a fibered product Lie algebroid over TM^{12}, the direct product Lie algebroid (denoted simply as $A^1 \times A^2$) of A^1 and A^2; it is straightforward to check that it is indeed a direct product in the category of Lie algebroids.

Proposition 2.3. *Consider morphisms of Lie algebroids $\phi_{1,2} : A^{1,2} \to B$ over $f_{1,2} : M^{1,2} \to N$, such that $\phi_1 \pitchfork \phi_2$. Then, the vector bundle fibered product $A^1 {}_{\phi_1}\!\times_{\phi_2} A^2 \to M^1 {}_{f_1}\!\times_{f_2} M^2$ carries a unique Lie algebroid structure making it a Lie subalgebroid of the direct product $A^1 \times A^2$, thus a fibered product in the category of Lie algebroids.*

[7]The existence of fibered product of Lie algebroids under the natural transversality conditions was stated without proof in [HM].

Proof. Denote with M the fibered product $M^1 {}_{f_1}\!\times_{f_2} M^2$ and with $p_{1,2} : M \to M^{1,2}$ the restrictions of the projections on the first and second component. Transversality for f_1 and f_2 implies $\delta := (f_1 \times f_2)|_M = p_{1,2} \circ f_{1,2}$ being submersive to the diagonal Δ_N; then, upon identifying N with Δ_N, B can be pulled back to a Lie algebroid over M. The pullback Lie algebroids $p^{++}_{1,2} A^{1,2} \to M$ also exist and there is a fibered product Lie algebroid $p^{++}_1 A^1 {}_{\psi_1}\!\times_{\psi_2} p^{++}_2 A^2$ over $\delta^{++}B$, where the morphisms $\psi_{1,2}$ are the unique obtained factorizing the compositions of $\phi_{1,2}$ with the inductors $p^{++}_{1,2} A^{1,2} \to A^{1,2}$ along the identity of M:

$$
\begin{tikzcd}
p^{++}_{1,2}A^{1,2} \arrow[r] \arrow[d] \arrow[drrr, "\psi_{1,2}"] & A^{1,2} \arrow[r, "\phi_{1,2}"] \arrow[d] & B \arrow[d] & \\
M \arrow[r] \arrow[drrr, equal] & M^{1,2} \arrow[r] & N & \delta^{++}B \arrow[ul] \arrow[d] \\
& & & M \arrow[ul]
\end{tikzcd}
.
$$

Note that $A^1 {}_{\phi_1}\!\times_{\phi_2} A^2$ and $p^{++}_1 A^1 {}_{\psi_1}\!\times_{\psi_2} p^{++}_2 A^2$ coincide and are smooth manifolds due to transversality of $\phi_{1,2}$, thus $A^1 {}_{\phi_1}\!\times_{\phi_2} A^2$ inherits a Lie algebroid structure from $p^{++}_1 A^1 {}_{\psi_1}\!\times_{\psi_2} p^{++}_2 A^2$. Next we show that there exists a unique morphism of Lie algebroids $\chi : A^1 {}_{\phi_1}\!\times_{\phi_2} A^2 \to A^1 \times A^2$ filling the diagram

$$
\begin{tikzcd}
A^1 {}_{\phi_1}\!\times_{\phi_2} A^2 \arrow[r] \arrow[dd] \arrow[drr, "\chi"] & p^{++}_2 A^2 \arrow[dd] \arrow[drr, "\chi_2"] & & \\
& & A^1 \times A^2 \arrow[r] \arrow[dd] & \mathrm{pr}^{++}_2 A^2 \arrow[dd] \\
p^{++}_1 A^1 \arrow[r] \arrow[drr, "\chi_1"] & TM \arrow[drr, hook] & & \\
& & \mathrm{pr}^{++}_1 A^1 \arrow[r] & TM^{12}
\end{tikzcd}
.
$$

The maps to TM and TM^{12} are anchors and $\chi_{1,2}$

$$
\begin{tikzcd}
p^{++}_{1,2}A^{1,2} \arrow[r] \arrow[d] \arrow[drr, "\chi_{1,2}"] & A^{1,2} \arrow[d] & \\
M \arrow[r] \arrow[drr, hook] & M^{1,2} & \mathrm{pr}^{++}_{1,2}A^{1,2} \arrow[ul] \arrow[d] \\
& & M^{12} \arrow[ul]
\end{tikzcd}
$$

are the unique morphisms factorizing the inductors of $p^{++}_{1,2} A^{1,2}$ through the inductors of $\mathrm{pr}^{++}_{1,2} A^{1,2}$ along the natural inclusion $M \hookrightarrow M^{12}$ of the fibered product base manifold. From the anchor compatibility condition for $\chi_{1,2}$ and the commutativity of last diagram, $\chi_{1,2}$ must be the canonical inclusions. Moreover, post composing $\chi_{1,2}$ with the inductors $\mathrm{pr}^{++}_{1,2} A^{1,2} \to A^{1,2}$, yields morphisms of Lie algebroids

$p_{1,2}^{++}A^{1,2} \to A^{1,2}$; by universality of $A^1 \times A^2$, χ exists and is unique. Thus, χ is the canonical inclusion, due to the commutativity of the defining diagram. To check universality of $A^1\, {}_{\phi_1}\!\times_{\phi_2} A^2$ is now straightforward. $\square$

It is then clear that the Lie algebroid of a fibered product $\mathcal{G}^1\, {}_{\varphi_1}\!\times_{\varphi_2} \mathcal{G}^2$ Lie groupoid coincides, as a vector bundle, with the fibered product $A^1\, {}_{\phi_1}\!\times_{\phi_2} A^2$ for the induced morphisms $\phi_{1,2}$ (the existence of such being assured from the source transversality condition: $A^1\, {}_{\phi_1}\!\times_{\phi_2} A^2 = \mathsf{ker}_{M^1\, {}_{f_1}\!\times_{f_2} M^2}(\phi_1 - \phi_2)$, where the bundle map $(\phi_1 - \phi_2) : A^1 \times A^2|_{M^1\, {}_{f_1}\!\times_{f_2} M^2} \to B$ has maximal rank); the induced Lie algebroid structure is then that of a fibered product, being $\mathcal{G}^1\, {}_{\varphi_1}\!\times_{\varphi_2} \mathcal{G}^2 \subset \mathcal{G}^1 \times \mathcal{G}^2$ a Lie subgroupoid, by uniqueness.

Example 2.4. Consider a Lie algebroid $A \to M$ and a smooth map $f : N \to M$. The requirement $\mathrm{d}\,f \pitchfork \rho$ is precisely the transversality condition for the pullback algebroid $f^{++}A$ to exist; in this case the fibred product Lie algebroid $TN\, {}_{\mathrm{d}\,f}\!\times_{\rho} A \to \Gamma(f)$ also exists and coincides with $f^{++}A$ up to the identification $\Gamma(f) \simeq N$.

3 A functorial approach to the integration of $\mathcal{LA}$-groupoids

Following Pradines [P], a differentiable graph is a pair of manifolds (Γ, M), endowed with surjective submersions $\alpha, \beta : \Gamma \to M$. We shall say that a graph is unital, if there is an injective immersion $\varepsilon : M \to \Gamma$, for which both α and β are left inverses, resp. invertible, if there is a diffeomorphism $\iota : \Gamma \to \Gamma$, such that $\iota^2 = \mathsf{id}_\Gamma$, $\alpha \circ \iota = \beta$ and $\iota \circ \varepsilon = \varepsilon$. Namely, an invertible unital graph is a “Lie groupoid without a multiplication”. For any differentiable graph, each nerve

$$\Gamma^{(n)} = \Gamma\, {}_{\alpha}\!\times_{\beta \circ \mathrm{p}_1} \Gamma^{(n-1)} \subset \Gamma^{\times n} \quad , \quad n > 1 \quad ,$$

is a smooth submanifold (conventionally, $\Gamma^{(1)} = \Gamma$ and $\Gamma^{(0)} = M$).

Let $(\Omega, A; \mathcal{G}, M)$ be an $\mathcal{LA}$-groupoid; if $\Omega \to \mathcal{G}$ is integrable, so is the Lie subalgebroid $A \to M$. Denote with $\Xi \rightrightarrows \mathcal{G}$ and $\mathcal{A} \rightrightarrows M$ the source 1-connected integrations. The top groupoid structural maps $\hat{\mathrm{s}}$, $\hat{\mathrm{t}} : \Omega \to A$, $\hat{\varepsilon} : A \to \Omega$ and $\hat{\iota} : \Omega \to \Omega$ are morphisms of Lie algebroids and integrate uniquely to morphisms of Lie groupoids s_V, $\mathrm{t}_V : \Xi \to \mathcal{A}$, $\varepsilon_V : \mathcal{A} \to \Xi$ and $\iota_V : \Xi \to \Xi$. The compatibility conditions $\hat{\mathrm{s}} \circ \hat{\varepsilon} = \mathsf{id}_A$, $\hat{\mathrm{t}} \circ \hat{\varepsilon} = \mathsf{id}_A$, $\hat{\mathrm{s}} \circ \hat{\iota} = \hat{\mathrm{t}}$, $\hat{\iota} \circ \hat{\iota} = \mathsf{id}_\Omega$ and $\hat{\iota} \circ \hat{\varepsilon} = \hat{\varepsilon}$ are diagrams of Lie algebroid morphisms and integrate to analogous relations for ι_V, s_V, t_V and ε_V. Then ι_V is a diffeomorphism (being the inverse to itself), ε_V is injective (it has left inverses), s_V and t_V are surjective (being left inverses to ε_V). Actually, ε_V is an immersion, s_V and t_V are submersive on an open neighbourhood of $\mathcal{G}$; this can be seen taking the tangent diagrams to $\mathrm{s}_V \circ \varepsilon_V = \mathsf{id}_{\mathcal{G}}$ and $\mathrm{t}_V \circ \varepsilon_V = \mathsf{id}_{\mathcal{G}}$. The following fact is obvious for Lie groups, by equivariance under right translations.

Lemma 3.1. *Let $\varphi : \mathcal{G} \to \mathcal{G}'$ be a morphism of Lie groupoids over a submersive base map $f : M \to M'$, such that φ is submersive on an open neighbourhood of M; then φ is submersive on all of $\mathcal{G}$.*

Proof. For any $g \in \mathcal{G}$, define $\sigma_g := T_g\mathcal{G} \cap \mathrm{d}\varphi^{-1} T^{\mathrm{s}'}_{\varphi(g)}\mathcal{G}'$, fix an arbitrary complement κ_g and note that $T^{\mathrm{s}}_g\mathcal{G}$, $\mathsf{ker}_g \mathrm{d}\varphi \subset \sigma_g$; thus $\kappa_g \simeq \mathrm{d}\varphi\, \kappa_g$, $T^{\mathrm{s}'}_{\varphi(g)}\mathcal{G}' \cap \mathrm{d}\varphi\, \kappa_g = \{0\}$ and it is sufficient to show: (1) $\mathrm{d}\varphi : T^{\mathrm{s}}_g\mathcal{G} \to T^{\mathrm{s}'}_g\mathcal{G}$ is surjective, (2) $\mathsf{dim}\, \kappa_g = \mathsf{dim}\, M'$. To see (1), for any $\delta' \in T^{\mathrm{s}'}_{\varphi(g)}\mathcal{G}'$ pick a $\delta_o \in T_{\varepsilon(\mathrm{t}(g))}\mathcal{G} \cap \mathrm{d}\varphi^{-1}(\mathrm{d}r^{-1}_{\varphi(g)}\delta')$, then $\delta := \mathrm{d}r_g(\delta_o - \mathrm{d}\varepsilon\, \mathrm{d}s\, \delta_o) \in T^{\mathrm{s}}_g\mathcal{G}$ maps on δ', by equivariance under right translation of the restriction of $\mathrm{d}\varphi$ to the subbundle of vectors tangent to the source fibres. To see (2), consider that, picking a local bisection Σ_g of $\mathcal{G}$ through g, yields a splitting $\sigma_g = T^{\mathrm{s}}_g\mathcal{G} \oplus \mathrm{d}\Sigma_g \mathsf{ker}_{\mathrm{s}(g)}\mathrm{d}f$ and count dimensions. Alternatively, it is sufficient to check (1) and apply the 5-lemma to the diagram for ϕ, f and the short exact sequences associated to the tangent source maps. □

It follows that s_V and t_V are surjective submersions and $(\Xi, \mathcal{A})$ is an invertible unital graph in the category of Lie groupoids. Moreover, the top vertical nerves of the graph

$$\Xi^{(n)}_V = \Xi\ {}_{\mathrm{s}_V}\!\times_{\mathrm{t}_V \circ \mathrm{pr}_1} \Xi^{(n-1)}_V \subset \Xi^{\times n} \qquad , \qquad n > 1 \qquad ,$$

($\Xi^{(1)}_V = \Xi$, $\Xi^{(0)}_V = \mathcal{A}$) can be inductively endowed with Lie groupoid structures (transversality conditions are met) over the side vertical nerves $\mathcal{G}^{(n)}$, the groupoid being induced by $\Xi \rightrightarrows \mathcal{G}$.

In general, the following holds.

Theorem 3.2. *Let $A^{1,2}$ and B be integrable Lie algebroids; consider morphisms of Lie algebroids $\phi_{1,2} : A^{1,2} \to B$, such that transversality conditions of proposition* (2.3) *hold and denote with $\varphi_{1,2} : \mathcal{G}^{1,2} \to \mathcal{H}$ any integrations. Then, the fibred product Lie groupoid $\mathcal{G}^1\ {}_{\varphi_1}\!\times_{\varphi_2} \mathcal{G}^2$ exists.*

Proof. With the same notations as in §2, consider that transversality for $\phi_{1,2}$ is equivalent to the source transversality condition (2.1) for $\varphi_{1,2}$ along M; to see that the condition holds off the base manifold, use a right translation argument as in lemma (3.1). It is now sufficient to prove transversality for $\varphi_{1,2}$, i.e to showing that $(\mathrm{d}\varphi_1 + \mathrm{d}\varphi_2) : T_{g_1}\mathcal{G}^1 \oplus T_{g_2}\mathcal{G}^2 \to T_h\mathcal{H}$ is surjective for all $(g_1, g_2) \in \mathcal{G}^1\ {}_{\varphi_1}\!\times_{\varphi_2} \mathcal{G}^2$; clearly the property holds sourcewise, thus, set

$$\sigma_{(g_1,g_2)} := T_{g_1}\mathcal{G}^1 \oplus T_{g_2}\mathcal{G}^2 \cap (\mathrm{d}\varphi_1 + \mathrm{d}\varphi_2)^{-1} T^{\mathrm{s}}_h\mathcal{H} \qquad .$$

As in the proof of lemma (3.1), $\mathsf{ker}_{(g_1,g_2)}(\mathrm{d}\varphi_1 + \mathrm{d}\varphi_2)$, $T^{\mathrm{s}^1}_{g_1}\mathcal{G}^1 \oplus T^{\mathrm{s}^2}_{g_2}\mathcal{G}^2 \subset \sigma_{(g_1,g_2)}$ and for any choice of bisections $\Sigma^{1,2}_{g_{1,2}}$ of $\mathcal{G}^{1,2}$, there is a splitting

$$\sigma_{(g_1,g_2)} = (T^{\mathrm{s}^1}_{g_1}\mathcal{G}^1 \oplus T^{\mathrm{s}^2}_{g_2}\mathcal{G}^2) \oplus (\mathrm{d}\Sigma^1_{g_1} \oplus \mathrm{d}\Sigma^2_{g_2}) \mathsf{ker}_{(g_1,g_2)}(\mathrm{d}f_1 + \mathrm{d}f_2) \qquad ;$$

to conclude the proof, count dimensions, using transversality of the base maps $f_{1,2}$, to show that $(\mathrm{d}\varphi_1 + \mathrm{d}\varphi_2)$ maps any complement of $\sigma_{(g_1,g_2)}$ to a complement of $T^{\mathrm{s}}_h\mathcal{H}$. As in lemma (3.1) one can replace the last part of this proof applying the 5-lemma to the suitable exact commuting diagram. $\square$

Actually, the graph Ξ carries more structure. Let $\mathbb{S}_\Xi$ denote the double source map $(\mathrm{s}_H, \mathrm{s}_V) : \Xi \to \mathcal{G}\ {}_{\mathrm{s}_h}\!\times_{\mathrm{s}_v} \mathcal{A}$; submersivity for $\mathbb{S}_\Xi$ is equivalent to fibrewise surjectivity of the restriction $\mathrm{ds}_V|_{T^{\mathrm{s}_H}\Xi} : T^{\mathrm{s}_H}\Xi \to T^{\mathrm{s}_h}\mathcal{A}$, which can be seen to hold by the usual right translation argument and fibrewise surjectivity of $\hat{\mathrm{s}} : \Omega \to A$, that is, surjectivity of the double source map \$. Then $(\Xi, \mathcal{G}, \mathcal{A}, M)$ fails to be a double Lie groupoid only for lacking a top vertical multiplication. Note however, that the Lie algebroids of the top vertical nerves are the nerves

$$\widehat{\Omega}^{(n)} = \Omega\ {}_{\hat{\mathrm{s}}}\!\times_{\hat{\mathrm{t}}\circ\mathrm{pr}_1} \widehat{\Omega}^{(n-1)} \subset \Omega^{\times n} \qquad , \qquad n > 1$$

of $\Omega \rightrightarrows A$ and the associativity diagram

$$\begin{tikzcd}
\widehat{\Omega}^{(3)}_V \arrow[r, "\mathrm{id}_\Omega\times\hat{\mu}"] \arrow[d, "\hat{\mu}\times\mathrm{id}_\Omega"'] & \widehat{\Omega}^{(2)}_V \arrow[d, "\hat{\mu}"] \\
\widehat{\Omega}^{(2)}_V \arrow[r, "\hat{\mu}"'] & \Omega
\end{tikzcd}$$

commutes in the category of Lie algebroids; thus, whenever the second top vertical nerve of Ξ is source 1-connected, $\hat{\mu}$ integrates to a morphism $\mu_V : \Xi^{(2)}_V \to \Xi$ and the diagram

$$\begin{tikzcd}
\Xi^{(3)}_V \arrow[r, "\mathrm{id}_\Xi\times\mu_V"] \arrow[d, "\mu_V\times\mathrm{id}_\Xi"'] & \Xi^{(2)}_V \arrow[d, "\mu_V"] \\
\Xi^{(2)}_V \arrow[r, "\mu_V"'] & \Xi
\end{tikzcd}$$

commutes in the category of Lie groupoids, provided the third top vertical nerve is source connected. To see this, consider that there is a similar *commuting* diagram, where $\Xi^{(3)}_V$ is replaced by its source 1-connected cover; commutativity of the associativity diagram follows by diagram chasing due to surjectivity of the covering morphism. Compatibility of μ_V with the graph over $(\Xi, \mathcal{A})$ can be easily shown by integrating the suitable compatibility diagrams for $\hat{\mu}$ to be a groupoid multiplication.

Under source connectivity assumptions on the second and third top vertical nerves, also integrability results for morphisms and sub-objects follow in the same fashion.

Even though no counter-examples are known to us, it seems unlikely for $\Xi^{(2,3)}_V$ to be source (1-)connected in general, for the vertically source 1-connected graph

Ξ of an $\mathcal{LA}$-groupoid. A sufficient condition for that to hold is:

for each $g \in \mathcal{G}$, the restriction $\mathrm{s}_V : {\mathrm{s}_H}^{-1}(g) \to \mathrm{s}_h^{-1}(\mathrm{s}_v(g))$ has the 1-homotopy lifting property.

Last requirement can be reduced to a lifting condition with respect to post composition with $\hat{\mathrm{s}}$ for A-paths, resp. A-homotopies, to Ω-paths, resp. Ω-homotopies (regarding Ω as a Lie algebroid over $\mathcal{G}$), involving only the $\mathcal{LA}$-groupoid-data. Recall from [CF1], that, for any Lie algebroid $(A \to M, [\,,\,], \rho)$, an A-path is a $\mathcal{C}^1$ map $\alpha : I \to A$, over a base path $X : I \to M$, such that $\mathrm{d}X = (\rho \circ X)\, \alpha$, namely a morphism of Lie algebroids $TI \to A$. An A-homotopy h from an A-path α_+ to an A-path α_- is a $\mathcal{C}^1$ morphism of Lie algebroids $TI^{\times 2} \to A$, satisfying suitable boundary conditions. Regarding h as a 1-form taking values in the pullback Y^+A, for the base map Y, the anchor compatibility condition is $\mathrm{d}Y = (\rho \circ Y)\, h$, while the bracket compatibility takes the form of a Maurer-Cartan equation

$$\mathrm{D}\, h + \frac{1}{2}[h \stackrel{\wedge}{,} h] = 0 \qquad ,$$

where D is the covariant derivative of the pullback of an arbitrary connection ∇ for A and $[\theta_+ \stackrel{\wedge}{,} \theta_-]$ is the contraction $-\iota_{\theta_+ \wedge \theta_-} Y^+ \tau^\nabla$ with the pullback of the torsion tensor, $\theta_\pm \in \Omega^1(Y^+A)$. The boundary conditions are $\iota^*_{\partial^\pm_{\mathrm{H}}} h = \alpha_\pm$ and $\iota^*_{\partial^\pm_{\mathrm{V}}} h = 0$, where

$$\partial^-_{\mathrm{H}(,\mathrm{V})} = \{(x,y) | x(y) = 0\} \quad \text{and} \quad \partial^+_{\mathrm{H}(,\mathrm{V})} = \{(x,y) | x(y) = 1\} \qquad .$$

Assume A is integrable and $\mathcal{G}$ the source 1-connected integration; then, the space of class $\mathcal{C}^2$ $\mathcal{G}$-paths and of A-paths are homeomorphic for the natural Banach topologies, the homeomorphism $\mathcal{G}$*-paths* $\to$ A*-paths* being given by the "right derivative" $\delta\, \Xi(t) = \mathrm{d}r^{-1}_{\Xi(t)} \dot{\Xi}(t)$, $t \in I$. Under such correspondence homotopy of $\mathcal{G}$-paths within the source fibers and relative to the endpoints, viz. $\mathcal{G}$-homotopy, translates precisely into A-homotopy.

We are ready to state our conditions.

Proposition 3.3. *With the same notations as above: all the nerves $\Xi_V^{(n)}$, $n \in \mathbb{N}$, of Ξ are:*

i) Source connected, if $\hat{\mathrm{s}}$ has the 0-$\mathcal{LA}$-homotopy lifting property, *i.e. for any Ω-path ξ and A-path α, which is A-homotopic to $\hat{\mathrm{s}} \circ \xi$, there exists a Ω-path ξ', which is Ω-homotopic to ξ and satisfies $\hat{\mathrm{s}} \circ \xi' = \alpha$, resp.*

ii) Source 1-connected, if $\hat{\mathrm{s}}$ has the 1-$\mathcal{LA}$-homotopy lifting property, *i.e. for any Ω-path ξ, which is Ω-homotopic to the constant Ω-path $\xi_o \equiv 0^\Omega_{\mathrm{Pr}(\xi(0))}$, and A-homotopy h from $\alpha := \hat{\mathrm{s}} \circ \xi$ to the constant A-path $\alpha_o \equiv 0^A_{\mathrm{pr}(\alpha(0))}$, there exists a Ω-homotopy $\hat{h}$ from ξ to ξ_o, such that $\hat{\mathrm{s}} \circ \hat{h} = h$.*

Proof. We shall prove the statement for $n = 2$, the argument extends easily by induction. (i) For any $x_\pm \in \Xi_V^{(2)}$ we have to find Ξ-paths $\gamma_\pm$, such that $\gamma_\pm(1) = x_\pm$ and $\mathrm{s}_V \circ \gamma_+ = \mathrm{t}_V \circ \gamma_-$. For any Ω-paths $\xi^\pm \in x_\pm$, $\alpha_+ := \hat{\mathrm{s}} \circ \xi^+$ is A-homotopic to $\alpha_- := \hat{\mathrm{t}} \circ \xi^-$ and we can assume $\alpha_- = \hat{\mathrm{s}} \circ \xi^+$. Define smooth families of Ω-paths $\xi_t^\pm(u) := t \cdot \xi^\pm(tu)$, u, $t \in I$; the corresponding (unique) families $\{\omega_t^\pm\}_{t\in I}$ of Ξ-paths are contained within the source fibers of $x_\pm$:

$$\mathrm{s}_H(\omega_t^\pm(u)) = \hat{\varepsilon}(\mathrm{Pr}(\xi_t^\pm(0))) = \hat{\varepsilon}(\mathrm{Pr}(\xi^\pm(0))) = \mathrm{s}_H(x_\pm) \qquad .$$

Thus, projecting to the quotient $\mathcal{W}(\Omega)$, yields Ξ-paths $\gamma_\pm(t) = [\xi_t^\pm]$, with $\gamma_\pm(1) = [\xi^\pm] = x_\pm$ and

$$\mathrm{s}_V(\gamma^+(t)) = [\mathrm{s}_V \circ \omega_t^+] = [\hat{\mathrm{s}} \circ \xi_t^+] = [\hat{\mathrm{t}} \circ \xi_t^-] = [\mathrm{t}_V \circ \omega_t^-] = \mathrm{t}_V(\gamma^-(t)) \qquad .$$

(ii) For any $\Xi_V^{(2)}$-loop (λ^+, λ^-), we have to find Ξ-homotopies $H_\pm$ from $\lambda^\pm$ to the constant Ξ-loops $\lambda_o^\pm \equiv \lambda^\pm(0)$, such that $\mathrm{s}_V \circ H^+ = \mathrm{t}_V \circ H^-$. We may as well regard $\lambda^\pm$ as Ξ-loops, satisfying $\mathrm{s}_V \circ \lambda^+ = \alpha = \mathrm{t}_V \circ \lambda^-$, for some $\mathcal{A}$-loop α; thus, the (unique) corresponding Ω-paths $\xi^\pm$ satisfy $\hat{\mathrm{s}} \circ \xi^+ = \hat{\alpha} = \hat{\mathrm{t}} \circ \xi^-$, for the A-path $\hat{\alpha}$ corresponding to α. Any Ω-homotopy h^- from ξ^- to the constant Ω-path ξ_o^- projects down to an A-homotopy $\hat{\mathrm{t}} \circ h^-$ from $\hat{\alpha}$ to the constant A-path $\hat{\alpha}_o$ and can be lifted along $\hat{\mathrm{s}}$ to an Ω-homotopy h^+ from ξ^+ to ξ_o^+, since Ξ is 1-connected and ξ^+ the Ω-path of a Ξ-loop, therefore Ω-homotopic to the constant Ω-path ξ_o^+. The Ξ-homotopies $H_\pm$ corresponding to $h^\pm$ are defined by $H^\pm(\varepsilon, t) = \tilde{h}_\pm(\varepsilon, t, 0, 0)$, ε, $t \in I$, where $\tilde{h}^\pm : I^{\times 2} \times I^{\times 2} \to \Xi$ are the morphisms of Lie groupoids integrating $h^\pm : TI^{\times 2} \to \Omega$. Therefore the condition $\mathrm{s}_V \circ H^+ = \mathrm{t}_V \circ H^-$ follows integrating $\hat{\mathrm{s}} \circ h^+ = \hat{\mathrm{t}} \circ h^-$ and $H_\pm$ are the desired homotopies. □

Therefore, we have:

Theorem 3.4. *Any $\mathcal{LA}$-groupoid Ω with integrable top Lie algebroid, whose associated integrating graph has source 1-connected top vertical nerve and source connected third top vertical nerve, is integrable to a horizontally source 1-connected double Lie groupoid. The connectivity assumptions hold when the $\mathcal{LA}$-homotopy lifting conditions of proposition* (3.3) *on Ω are met.*

Example 3.5. Consider the tangent prolongation $\mathcal{LA}$-groupoid of example 1.3. For any manifold N, TN-paths, resp. TN-homotopies, may be identified with ordinary paths in N, resp. homotopies in N relative to the endpoints. If the source map of $\mathcal{G}$ has the 1-homotopy lifting property[8], in particular when $\mathcal{G} \rightrightarrows M$ is a proper groupoid, our conditions are satisfied and the fundamental groupoid

[8]Necessary and sufficient conditions for a surjective submersion to be an n-fibration, $n \in \mathbb{N}$, can be found in [Mg].

$\Pi(\mathcal{G}) \rightrightarrows \mathcal{G}$ carries a further Lie groupoid structure over $\Pi(M)$, making

$$\begin{tikzcd} \Pi(\mathcal{G}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\ \Pi(M) \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

a double Lie groupoid; the top vertical multiplication is induced by multiplication of pointwise composable paths in $\mathcal{G}$. The double Lie groupoid of example (1.3) is clearly an integration of the same $\mathcal{LA}$-groupoid (even when $\mathcal{G}$ is not proper), which is, in general, not horizontally source 1-connected; if $\mathcal{G}$ itself is source 1-connected, since s is a submersion with 1-connected fibers, it is possible to apply proposition (7.1) of [G] to conclude that $\pi_{0,1}(\mathcal{G}) = \pi_{0,1}(M)$. Thus, when $\pi_{0,1}(M) = 0$, our approach produces the pair double groupoid of $\mathcal{G}$.

Consider any double Lie groupoid; since there is no natural way to to replace the horizontal Lie groupoids with the 1-connected covers of their source connected components without affecting the top vertical groupoid, in general, one should not expect to be able to integrate an $\mathcal{LA}$-groupoid to a horizontally source 1-connected double Lie groupoid. However, examples in which our approach applies should arise from proper actions of Lie groupoids and Lie algebroids; we shall study elsewhere less restrictive conditions to integrate $\hat{\mu}$ to a groupoid multiplication for $(\Xi, \mathcal{A})$ and discuss more interesting examples.

4 The case of an integrable Poisson groupoid

Consider a Poisson groupoid $(\mathcal{P}, \Pi) \rightrightarrows M$, with an integrable Poisson structure and the associated $\mathcal{LA}$-groupoid (as in §1); denote with $\mathcal{S}$ the source 1-connected symplectic groupoid integrating $(\mathcal{P}, \Pi)$ and with $(\overline{\mathcal{P}}, \overline{\Pi}) \rightrightarrows M$ the source 1-connected weak dual Poisson groupoid. The graph of $\mathcal{S}$ over $\overline{\mathcal{P}}$ is obtained integrating the structural maps of the graph underlying the cotangent prolongation groupoid $T^*\mathcal{P} \rightrightarrows A^*$. The next result follows specializing the results of last section.

Theorem 4.1. *Let $(\mathcal{P}, \Pi) \rightrightarrows M$ be a Poisson groupoid with integrable Poisson structure. Then, the graph*

$$\begin{tikzcd} \mathcal{S} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{P} \arrow[d, shift left] \arrow[d, shift right] \\ \overline{\mathcal{P}} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd} \tag{4.1}$$

integrating the $\mathcal{LA}$-groupoid associated with $\mathcal{P}$ is a symplectic double groupoid, whenever it has source 1-connected second vertical nerve and source connected third vertical nerve, this is the case when the $\mathcal{LA}$-homotopy lifting conditions of proposition (3.3) *are met. Moreover, $\mathcal{S}$ is a double of $\mathcal{P}$.*

Note that we make no connectivity assumptions on $\mathcal{P}$.

Proof. Under the connectivity assumptions, the procedure of §3 yields a horizontally source 1-connected double Lie groupoid on (4.1), whose top horizontal structure is the symplectic groupoid of $(\mathcal{P}, \Pi)$. Recall that the (ordinary) graph of the cotangent multiplication $\hat{\mu}$ is (by construction indeed [CDW])

$$\mathsf{\Gamma}(\hat{\mu}) = (\mathsf{id}_{T^*\mathcal{P}} \times \mathsf{id}_{T^*\mathcal{P}} \times -\mathsf{id}_{T^*\mathcal{P}})\, N^*\mathsf{\Gamma}(\mu) \subset T^*\mathcal{P}^{\times 3} \qquad ,$$

where μ is the multiplication of $\mathcal{P}$, therefore it is a Lagrangian submanifold with respect to the symplectic form $\omega_{can} \times \omega_{can} \times -\omega_{can}$. Moreover, $\mathsf{\Gamma}(\mu)$ is coisotropic in $(\mathcal{P}^{\times 3}, \Pi \times \Pi \times -\Pi)$, thus $\mathsf{\Gamma}(\hat{\mu}) \to \mathsf{\Gamma}(\mu)$ is canonically a Lie subalgebroid of $T^*\mathcal{P}^{\times 3} \to \mathcal{P}^{\times 3}$ for the Koszul bracket. From integrability of Π, it follows that $\mathsf{\Gamma}(\hat{\mu})$ integrates to an *immersed* Lagrangian [C] subgroupoid $\mathcal{L} \subset \mathcal{S}^{\times 3}$ (symplectic form with a minus sign on the third component). Note that the graph $\mathsf{\Gamma}(\varphi)$ of a morphism of Lie groupoids $\varphi : \mathcal{G} \to \mathcal{G}'$ is always a Lie subgroupoid of $\mathcal{G} \times \mathcal{G}'$ over the graph of its base map, isomorphic to the domain groupoid. Since the projection to the first and second factor $\mathsf{\Gamma}(\mu_V) \to \mathcal{S}_V^{(2)}$ is an isomorphism of Lie groupoids, $\mathsf{\Gamma}(\mu_V)$ is also source a 1-connected Lie groupoid, therefore coinciding with the Lie groupoid $\mathcal{L}$ of $\mathsf{\Gamma}(\hat{\mu})$, and a Lagrangian *Lie* subgroupoid of $\mathcal{S}^{\times 3}$. Hence $\mathcal{S}$ is a symplectic double groupoid.
Moreover, the unique Poisson structure Π' induced by $\mathcal{S}$ on $\overline{\mathcal{P}}$, makes $(\overline{\mathcal{P}}, \Pi') \rightrightarrows M$ a Poisson groupoid weakly dual to $(\mathcal{P}, \Pi) \rightrightarrows M$ and $\Pi' = \overline{\Pi}$, by uniqueness (theorem (1.6)); that is, $\mathcal{S}$ is a double for $(\mathcal{P}, \Pi) \rightrightarrows M$. □

In the proof of last theorem the fact that the second vertical nerve of the graph on $\mathcal{S}$ is 1-connected is not only essential to define the top horizontal multiplication, but also to show that it is compatible with the symplectic form. Note that $\mathcal{S} \rightrightarrows \overline{\mathcal{P}}$, will not be, in general, source 1-connected.

We conclude this note remarking that Lu and Weinstein's approach to the construction of the double of a Poisson group does not apply to Poisson groupoids, since it relies on the peculiar properties of the coadjoint action of a Poisson group on its dual, features lacking in the theory of Lie groupoids. Moreover the Drinfel'd doubles of Lie bialgebroids are no longer Lie algebroids. In fact, they can be characterized as double Lie algebroids (Mackenzie's double), or [LWX] Courant algebroids.

On the other hand, to any Courant algebroid, one can associate a tridimensional topological field theory, named *Courant sigma model* by Strobl [S]. The action functional of this theory was obtained by Ikeda as a deformation in the BRST antifield formalism of the abelian Chern-Simons theory coupled with a BF term, but it can also be understood as the AKSZ action for a symplectic Q-manifold of degree 2 [R].

The phase space of the classical version of this theory can always be described as a constrained Hamiltonian system. In the case of the Courant algebroid of a Lie bialgebroid (A, A^*), with integrable Lie algebroids, choosing the square as a source manifold and suitable boundary conditions, the reduced space, when a manifold, carries a symplectic form and two compatible differentiable (unital, invertible) graphs over the Poisson groupoid of (A, A^*) and its weak dual. Moreover the geometry of the source manifold allows to define two partial multiplications before taking the symplectic quotient and yields two compatible Lie groupoid multiplications on the reduced space, provided some obstruction related to the Π_2's of the leaves of A and A^* vanishes. The chance of integrating simultaneously a Poisson groupoid and its weak dual by taking a quotient of the reduced space of the Courant sigma model is under investigation in a joint work with Alberto Cattaneo.

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