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# An approach toward a finite-dimensional definition of twisted $K$-theory [1]

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**Abstract**

This is an expository account of the following result: we can construct a group by means of twisted $\mathbb{Z}_2$-graded vectorial bundles which is isomorphic to $K$-theory twisted by any degree three integral cohomology class.

## 1 Introduction

Topological $K$-theory admits a twisting by a degree three integral cohomology class. The resulting $K$-theory, known as *twisted $K$-theory* [2], has its origin in the works of Donovan-Karoubi [8] and Rosenberg [15], and has applications to D-brane charges [5, 12, 17], Verlinde algebras [9] and so on.

As is well-known, ordinary $K$-theory has definitions by means of:

(1) vector bundles;

(2) the $C^*$-algebra of continuous functions; and

(3) the space of Fredholm operators.

Twisted $K$-theory is usually defined by twisting the definitions (2) or (3). For a definition parallel to (1), there are the notions of *twisted vector bundles* ([13, 14, 16], see also [5, 8, 12]), and of *bundle gerbe $K$-modules* [4]. However, the definitions by means of these geometric objects are only valid for twisted $K$-theory whose "twisting", a third integral cohomology class, is of finite order: otherwise, there is no non-trivial such geometric object in finite-dimensions.

Toward a finite-dimensional definition of twisted $K$-theory valid for degree three integral cohomology classes of infinite order, we explain in this article Furuta's notion of generalized vector bundles [10], which we call *vectorial bundles*. We also explain a notion of finite dimensional approximation of Fredholm operators, which provides a linear version of the finite dimensional approximation of the monopole equations [11]. We can use these notions to construct a group and an isomorphism from $K$-theory twisted by any degree three integral cohomology class. The proof of the result is only outlined. The detailed treatment will be provided elsewhere.

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A possible application of the result above is to generalize the notions of *2-vector bundles* [3, 6]. A 2-vector bundle of rank 1 due to Brylinski [6] is a stack which reproduces the category of twisted vector bundles. Replacing the category of vector bundles by that of vectorial bundles, we can directly generalize the 2-vector bundles in [6]. Similarly, we can also generalize the 2-vector bundles due to Baas, Dundas and Rognes [3], which they studied in an approach to geometric realization of elliptic cohomology. It seems interesting to apply their study of 2-vector bundles to the generalization of 2-vector bundles made of vectorial bundles.

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# 2 Twisted $K$-theory and twisted vector bundles

In this section, we review the definition of twisted $K$-theory by means of the space of Fredholm operators, following [2]. We also review the notion of twisted vector bundles [4, 5, 8, 12, 13, 14, 16].

Unless otherwise mentioned, $X$ is a compact manifold through this article.

## 2.1 Twisted $K$-theory

The twisted $K$-theory we consider in this article is associated to a degree three integral cohomology class. To give the precise definition, we represent the class by a projective unitary bundle. Let $\mathcal{H}$ be a separable Hilbert space of infinite dimension, and $PU(\mathcal{H})$ the projective unitary group $PU(\mathcal{H}) = U(\mathcal{H})/U(1)$.

**Definition 2.1.** For a principal $PU(\mathcal{H})$-bundle over $X$, we define the *twisted $K$-group* $K_P(X)$ to be the fiberwise homotopy classes of sections of the associated bundle $P \times_{Ad} \mathcal{F}(\mathcal{H})$ over $X$, where $PU(\mathcal{H})$ acts on the space $\mathcal{F}(\mathcal{H})$ of Fredholm operators on $\mathcal{H}$ by adjoint.

In the above definition, the topologies on $PU(\mathcal{H})$ and $\mathcal{F}(\mathcal{H})$ are understood to come from the operator norm. Notice that we can also use the compact-open topology in the sense of [2].

If $P$ is a trivial bundle, then $K_P(X)$ is exactly the homotopy classes of continuous functions $X \to \mathcal{F}(\mathcal{H})$. Thus, in this case, we recover the $K$-group of $X$ by the well-known fact that $\mathcal{F}(\mathcal{H})$ is a classifying space of $K$-theory [1].

$PU(\mathcal{H})$-bundles over $X$ are classified by $H^3(X, \mathbb{Z})$: since $U(\mathcal{H})$ is contractible by Kuiper's theorem, $PU(\mathcal{H})$ is homotopy equivalent to the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$, so that the classifying space $BPU(\mathcal{H})$ is homotopy equivalent to

$K(\mathbb{Z},3)$. If $P$ and $P'$ are isomorphic $PU(\mathcal{H})$-bundles, then the twisted $K$-groups $K_P(X)$ and $K_{P'}(X)$ are also (non-canonically) isomorphic. So we often speak of "twisted $K$-theory twisted by a class in $H^3(X,\mathbb{Z})$".

We will call the cohomology class corresponding to $P$ the *Dixmier-Douady class*, and denote it by $\delta(P) \in H^3(X,\mathbb{Z})$. For later convenience, we recall here the construction of $\delta(P)$: take an open cover $\mathcal{U} = \{U_\alpha\}$ of $X$ so that:

- there are local sections $s_\alpha : U_\alpha \to P|_{U_\alpha}$;
- there are lifts $g_{\alpha\beta} : U_{\alpha\beta} \to U(\mathcal{H})$ of the transition functions $\overline{g}_{\alpha\beta} : U_{\alpha\beta} \to PU(\mathcal{H})$.

Here we write $U_{\alpha\beta}$ for the overlap $U_\alpha \cap U_\beta$, and the transition function is defined by the relation $s_\alpha \overline{g}_{\alpha\beta} = s_\beta$. Because of the cocycle condition for $\{\overline{g}_{\alpha\beta}\}$, we can find a map $c_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to U(1)$ such that $g_{\alpha\beta}g_{\beta\gamma} = c_{\alpha\beta\gamma}g_{\alpha\gamma}$. These maps comprise a Čech 2-cocycle $(c_{\alpha\beta\gamma}) \in \check{Z}^2(\mathcal{U}, \underline{U(1)})$ with its coefficients in the sheaf of germs of $U(1)$-valued functions, which represents $\delta(P)$ through the isomorphism $\check{H}^2(X, \underline{U(1)}) \cong H^3(X,\mathbb{Z})$.

## 2.2 Twisted vector bundles

For a $PU(\mathcal{H})$-bundle $P$ over $X$, a *twisted vector bundle* consists essentially of the data $(\mathcal{U}, E_\alpha, \phi_{\alpha\beta})$:

- an open cover $\mathcal{U} = \{U_\alpha\}$ of $X$;
- vector bundles $E_\alpha$ over $U_\alpha$;
- isomorphisms of vector bundles $\phi_{\alpha\beta} : E_\beta|_{U_{\alpha\beta}} \to E_\alpha|_{U_{\alpha\beta}}$ over $U_{\alpha\beta}$ satisfying the "twisted cocycle condition" on $U_{\alpha\beta\gamma}$:

$$\phi_{\alpha\beta}\phi_{\beta\gamma} = c_{\alpha\beta\gamma}\phi_{\alpha\gamma},$$

  where $c_{\alpha\beta\gamma}$ is as in the previous subsection.

In the rigorous definition of twisted vector bundles, we have to include the choices of the local sections $s_\alpha$ and the lifts $g_{\alpha\beta}$. Though it is crucial to specify these choices in considering isomorphism classes of twisted vector bundles, we omit them for simplicity.

The isomorphism classes of twisted vector bundles $\mathrm{Vect}_P(X)$ gives rise to a semi-group by the direct sum of local vector bundles. Let $K(\mathrm{Vect}_P(X))$ denote the group given by applying the Grothendieck construction to $\mathrm{Vect}_P(X)$. Then the following fact is known. (See [5, 8, 12, 13, 16].)

**Proposition 2.2.** *For a $PU(\mathcal{H})$-bundle whose Dixmier-Douady class $\delta(P)$ is of finite order, there exists an isomorphism:*

$$K_P(X) \longrightarrow K(\mathrm{Vect}_P(X)).$$

Instead of twisted vector bundles, we can use bundle gerbe $K$-modules to obtain an equivalent result [4, 7].

The rank of a twisted vector bundle is a multiple of the order of $\delta(P)$. This can be seen readily as follows. Suppose that a twisted vector bundle has a finite rank $r$. Taking the determinant of the twisted cocycle condition, we have:

$$\det\phi_{\alpha\beta}\det\phi_{\beta\gamma} = c^r_{\alpha\beta\gamma}\det\phi_{\alpha\gamma}.$$

Hence $(c^r_{\alpha\beta\gamma}) \in \check{Z}^2(\mathcal{U}, \underline{U(1)})$ is a coboundary and $r\delta(P) = 0$.

Because of the above remark, there are no no-trivial twisted vector bundles in the case where $\delta(P)$ is infinite order. So we cannot use twisted vector bundles of finite dimensions to realize $K_P(X)$ generally. In spite of this fact, collections of locally defined vector bundles seem to have the potential in defining $K_P(X)$ by means of finite-dimensional objects. An approach is to use the usual technique proving the isomorphism $K(X) \cong [X, \mathcal{F}(\mathcal{H})]$. In this approach, however, some complications prevent us from transparent management, in particular, in giving equivalence relation. The usage of Furuta's generalized vector bundle provides a more efficient approach, which we explain in the next section.

# 3 Furuta's generalized vector bundle

In this section, we explain a generalization of the notion of vector bundles introduced by M. Furuta [10]. We call the generalized vector bundles *vectorial bundles* for short. This notion is closely related to a finite-dimensional approximation of Fredholm operators. Applying these notions, we approach to our problem of defining twisted $K$-theory finite-dimensionally.

## 3.1 Approximation of a Fredholm operator

We begin with the simplest situation. A $\mathbb{Z}_2$-graded vectorial bundle over a single point is a pair $(E, h)$ consisting of:

- a $\mathbb{Z}_2$-graded Hermitian vector space $E = E^0 \oplus E^1$ of finite rank; and
- a Hermitian map $h : E \to E$ of degree 1.

By using a $\mathbb{Z}_2$-graded vectorial bundle over a point, we can approximate a single Fredholm operator as follows. Let $A : \mathcal{H} \to \mathcal{H}$ be a Fredholm operator. For simplicity, we assume the kernel or cokernel of $A$ is non-trivial We define the $\mathbb{Z}_2$-graded Hilbert space $\hat{\mathcal{H}}$ by $\hat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$, and the self-adjoint Fredholm operator $\hat{A} : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ of degree 1 by $\hat{A} = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$. By the assumption, the spectrum $\sigma(\hat{A}^2)$ of the non-negative operator $\hat{A}^2$ contains 0. Since $\hat{A}^2$ is also Fredholm, $0 \in \sigma(\hat{A}^2)$ is a discrete spectrum. Hence there is a positive number $\mu$ such that:

- $\mu \notin \sigma(\hat{A}^2)$;
- the subset $\sigma(\hat{A}^2) \cap [0, \mu)$ consists of a finite number of eigenvalues;
- the eigenspace $\mathrm{Ker}(\hat{A}^2 - \lambda)$ is finite dimensional for $\lambda \in \sigma(\hat{A}^2) \cap [0, \mu)$.

Let $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n < \mu$ be the distinct eigenvalues in $\sigma(\hat{A}^2) \cap [0, \mu)$. Then we have the following orthogonal decomposition of $\hat{\mathcal{H}}$:

$$\hat{\mathcal{H}} = (\hat{\mathcal{H}}, \hat{A})_{\lambda_1} \oplus (\hat{\mathcal{H}}, \hat{A})_{\lambda_2} \oplus \cdots \oplus (\hat{\mathcal{H}}, \hat{A})_{\lambda_n} \oplus \hat{\mathcal{H}}',$$

where $(\hat{\mathcal{H}}, \hat{A})_\lambda = \mathrm{Ker}(\hat{A}^2 - \lambda)$ is the eigenspace of $\hat{A}^2$ with its eigenvalue $\lambda$, and $\hat{\mathcal{H}}'$ is the orthogonal complement. Notice that $\hat{A}$ preserves each eigenspace as well as the orthogonal complement. More precisely, $\hat{A}$ restricts to the trivial map on $(\hat{\mathcal{H}}, \hat{A})_{\lambda_1} \cong \mathrm{Ker}A \oplus \ker A^*$, while $\hat{A}$ induces isomorphisms on $(\mathcal{H}, \hat{A})_{\lambda_i}$ for $i > 1$ and $\hat{\mathcal{H}}'$.

Now, cutting off the infinite-dimensional part $\hat{\mathcal{H}}'$, we define $E$ and $h$ by $E = \bigoplus_i (\hat{\mathcal{H}}, \hat{A})_{\lambda_i}$ and $h = \hat{A}|_E$. The pair $(E, h)$ is nothing but a $\mathbb{Z}_2$-graded vectorial bundle over a point.

As a finite-dimensional approximation of a single Fredholm operator, we obtained a $\mathbb{Z}_2$-graded vectorial bundle over a single point. As will be explained in Subsection 3.3, a similar approximation is possible for a family of Fredholm operators parameterized by $X$. The resulting object is a $\mathbb{Z}_2$-graded vectorial bundle over $X$.

## 3.2 Definition of vectorial bundle

We now introduce $\mathbb{Z}_2$-graded vectorial bundles:

**Definition 3.1.** Let $(\mathcal{U}, (E_\alpha, h_\alpha), \phi_{\alpha\beta})$ be the following data:

- an open cover $\mathcal{U} = \{U_\alpha\}$ of $X$;
- $\mathbb{Z}_2$-graded Hermitian vector bundles $E_\alpha$ over $U_\alpha$;
- Hermitian maps $h_\alpha : E_\alpha \to E_\alpha$ of degree 1;
- vector bundle maps $\phi_{\alpha\beta} : E_\beta|_{U_{\alpha\beta}} \to E_\alpha|_{U_{\alpha\beta}}$ of degree 0 over $U_{\alpha\beta}$ such that $h_\alpha \phi_{\alpha\beta} = \phi_{\alpha\beta} h_\beta$.

A $\mathbb{Z}_2$*-graded vectorial bundle* over $X$ is defined to be data $(\mathcal{U}, (E_\alpha, h_\alpha), \phi_{\alpha\beta})$ satisfying the following conditions:

$$\begin{aligned} \phi_{\alpha\alpha} &\doteq 1 && \text{on } U_\alpha, \\ \phi_{\alpha\beta}\phi_{\beta\gamma} &\doteq \phi_{\alpha\gamma} && \text{on } U_{\alpha\beta\gamma}. \end{aligned}$$

In the above definition, the symbol $\doteq$ stands for an equivalence relation. The meaning of the first condition $\phi_{\alpha\alpha} \doteq 1$ is as follows.

> For any point $x \in U_\alpha$, there are a neighborhood $V \subset U_\alpha$ of $x$ and a positive number $\mu$ such that: for all $y \in V$ and $v \in (E_\alpha, h_\alpha)_{y,<\mu}$ we have $\phi_{\alpha\alpha}(v) = v$.

Here $(E_\alpha, h_\alpha)_{y,<\mu}$ is the subspace in the fiber of $E_\alpha$ at $y$ given by the direct sum of eigenspaces of $(h_\alpha)_y^2$ whose eigenvalues are less than $\mu$:

$$(E_\alpha, h_\alpha)_{y,<\mu} = \bigoplus_{\lambda<\mu} \mathrm{Ker}\left((h_\alpha)_y^2 - \lambda\right) = \bigoplus_{\lambda<\mu} \left\{v \in (E_\alpha)_y \middle|\ (h_\alpha)_y^2 v = \lambda v\right\}.$$

The meaning of the second condition $\phi_{\alpha\beta}\phi_{\beta\gamma} \doteq \phi_{\alpha\gamma}$ is now obvious.

**Definition 3.2.** Let $\mathbb{E} = (\mathcal{U}, (E_\alpha, h_\alpha), \phi_{\alpha\beta})$ and $\mathbb{E}' = (\mathcal{U}, (E'_\alpha, h'_\alpha), \phi'_{\alpha\beta})$ be $\mathbb{Z}_2$-graded vectorial bundles over $X$.

(a) A set $(f_\alpha)$ of vector bundle maps $f_\alpha : E_\alpha \to E'_\alpha$ of degree 0 such that $f_\alpha h_\alpha = h'_\alpha f_\alpha$ on $U_\alpha$ is said to be a *homomorphism* from $\mathbb{E}$ to $\mathbb{E}'$, if we have $f_\alpha \phi_{\alpha\beta} \doteq \phi'_{\alpha\beta} f_\beta$ on $U_{\alpha\beta}$.

(b) A homomorphism $(f_\alpha) : \mathbb{E} \to \mathbb{E}'$ is said to be an *isomorphism*, if there exists a homomorphism $(f'_\alpha) : \mathbb{E}' \to \mathbb{E}$ such that $f_\alpha f'_\alpha \doteq 1$ and $f'_\alpha f_\alpha \doteq 1$ on $U_\alpha$.

In the above definition of homomorphism, $\mathbb{E}$ and $\mathbb{E}'$ share the same open cover. In the case where they have different open covers $\mathcal{U}$ and $\mathcal{U}'$ respectively, it suffices to take a common refinement of $\mathcal{U}$ and $\mathcal{U}'$.

**Definition 3.3.** A *homotopy* between $\mathbb{Z}_2$-graded vectorial bundles $\mathbb{E}$ and $\mathbb{E}'$ over $X$ is defined to be a $\mathbb{Z}_2$-graded vectorial bundle $\tilde{\mathbb{E}}$ over $X \times [0, 1]$ such that $\mathbb{E}$ and $\mathbb{E}'$ are isomorphic to $\tilde{\mathbb{E}}|_{X\times\{0\}}$ and $\tilde{\mathbb{E}}|_{X\times\{1\}}$, respectively.

We write $KF(X)$ for the set of homotopy classes of isomorphism classes of $\mathbb{Z}_2$-graded vectorial bundles. The set $KF(X)$ gives rise to a group by means of the direct sum of vector bundles given locally.

A $\mathbb{Z}_2$-graded (ordinary) vector bundle $E$ gives an example of a $\mathbb{Z}_2$-graded vectorial bundle by setting $\mathcal{U} = \{X\}$ and $h = 0$. This construction induces a well-defined homomorphism $K(X) \to KF(X)$. In [10], Furuta proved:

**Proposition 3.4.** *The homomorphism $K(X) \to KF(X)$ is an isomorphism.*

## 3.3 Approximation of a family of Fredholm operators

As a family version of the construction in Subsection 3.1, we can show:

**Lemma 3.5.** *Let $A = \{A_x\} : X \to \mathcal{F}(\mathcal{H})$ be a continuous map. For any point $p \in X$, there are a neighborhood $U_p$ of $p$ and a positive number $\mu_p$ such that the following family of vector spaces gives rise to a vector bundle over $U_p$:*

$$\bigcup_{x\in U_p} (\hat{\mathcal{H}}, \hat{A}_x)_{<\mu_p} = \bigcup_{x\in U_p} \bigoplus_{\lambda<\mu_p} \mathrm{Ker}(\hat{A}_x^2 - \lambda).$$

A key to this lemma is that eigenvalues of $A_x$ is continuous in $x$.

By means of the lemma, the family of Fredholm operators $A : X \to \mathcal{F}(\mathcal{H})$ yields a $\mathbb{Z}_2$-graded vectorial bundle $(\{U_p\}_{p\in X}, (E_{U_p}, h_{U_p}), \phi_{U_pU_q})$, where the $\mathbb{Z}_2$-graded vector bundle $E_{U_p}$ is that in Lemma 3.5, the Hermitian map $h_{U_p}$ is given by restricting the Fredholm operator: $h_{U_p}|_x = \hat{A}_x|_{E_{U_p}}$, and the map of vector bundles $\phi_{U_pU_q} : E_{U_q} \to E_{U_p}$ is the following composition of the natural inclusion and the orthogonal projection:

$$\bigcup_{x\in U_p\cap U_q} (\hat{\mathcal{H}}, \hat{A}_x)_{<\mu_q} \longrightarrow (U_p \cap U_q) \times \hat{\mathcal{H}} \longrightarrow \bigcup_{x\in U_p\cap U_q} (\hat{\mathcal{H}}, \hat{A}_x)_{<\mu_p}.$$

The construction above induces a well-defined homomorphism

$$\alpha : \ [X, \mathcal{F}(\mathcal{H})] \longrightarrow KF(X).$$

This homomorphism is compatible with the isomorphism ind : $[X, \mathcal{F}(\mathcal{H})] \to K(X)$ in [1]. Namely, the following diagram is commutative:

$$\begin{array}{ccc} [X, \mathcal{F}(\mathcal{H})] & = & [X, \mathcal{F}(\mathcal{H})] \\ {\scriptstyle \mathrm{ind}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ K(X) & \longrightarrow & KF(X). \end{array}$$

The compatibility follows from the fact that one can realize any vector bundle $E \to X$ as $E = \bigcup_{x\in X} \mathrm{Ker}\hat{A}_x$ by taking $A : X \to \mathcal{F}(\mathcal{H})$ such that $\sigma(\hat{A}_x^2) = \{0, 1\}$. (See the proof of the surjectivity of ind in [1].)

## 3.4 Twisted vectorial bundle and twisted $K$-theory

We now apply vectorial bundles and finite dimensional approximations explained so far to twisted $K$-theory.

Recall that twisted vector bundles are defined by "twisting" the ordinary cocycle condition for vector bundles. In a similar way, for a $PU(\mathcal{H})$-bundle $P$, we define a *twisted $\mathbb{Z}_2$-graded vectorial bundle* by replacing the "cocycle condition" $\phi_{\alpha\beta}\phi_{\beta\gamma} \doteq \phi_{\alpha\gamma}$ in Definition 3.1 by the "twisted cocycle condition":

$$\phi_{\alpha\beta}\phi_{\beta\gamma} \doteq c_{\alpha\beta\gamma}\phi_{\alpha\gamma}.$$

A twisted $\mathbb{Z}_2$-graded vectorial bundle can be constructed from a section $\mathbb{A} : X \to P \times_{Ad} \mathcal{F}(\mathcal{H})$. The section gives a set of maps $\{A_p : W_p \to \mathcal{F}(\mathcal{H})\}_{p\in X}$ such that $A_p = g_{pq}A_qg_{pq}^{-1}$, where $W_p$ is an open set containing $p$ and $g_{pq} : W_p \cap W_q \to U(\mathcal{H})$ is a lift of transition function of $P$. Now, we use Lemma 3.5 to define a Hermitian vector bundle over $U_p \subset W_p$ by $E_{U_p} = \bigcup_{x\in U_p}(\hat{\mathcal{H}}, (\hat{A}_p)_x)_{<\mu_p}$. The map $A_p$ also defines a Hermitian map $h_{U_p}$ on $E_{U_p}$ by restriction. If we define $\phi_{U_pU_q} : E_{U_q} \to E_{U_p}$ by the following composition:

$$\bigcup_{x \in U_{pq}} (\hat{\mathcal{H}}, (\hat{A}_q)_x)_{<\mu_q} \to U_{pq} \times \hat{\mathcal{H}} \overset{\mathrm{id}\times g_{pq}}{\longrightarrow} U_{pq} \times \hat{\mathcal{H}} \to \bigcup_{x \in U_{pq}} (\hat{\mathcal{H}}, (\hat{A}_p)_x)_{<\mu_p},$$

then $(\{U_p\}, (E_{U_p}, h_{U_p}), \phi_{U_pU_q})$ is a twisted $\mathbb{Z}_2$-graded vectorial bundle.

Introducing isomorphisms and homotopies in a similar way, we obtain the group $KF_P(X)$ of homotopy classes of isomorphism classes of twisted $\mathbb{Z}_2$-graded vectorial bundles. The above construction of twisted vectorial bundles induces the well-defined homomorphism

$$\alpha : \ K_P(X) \longrightarrow KF_P(X).$$

Since this map generalizes $\alpha : [X, \mathcal{F}(\mathcal{H})] \to KF(X)$, it is reasonable to expect that $\alpha$ gives rise to an isomorphism. In fact, we have:

**Theorem 3.6.** *For any $PU(\mathcal{H})$-bundle $P$ over a compact manifold $X$, the homomorphism $\alpha : K_P(X) \longrightarrow KF_P(X)$ is bijective.*

We sketch the proof of this result in the next subsection.

## 3.5 Sketch of the proof of Theorem 3.6

The fundamental idea to prove Theorem 3.6 is to construct a kind of generalized cohomology theory on CW complexes by means of $KF_P(X)$.

As is known [2, 7], the twisted $K$-group $K_P(X)$ fits into a certain generalized cohomology theory $\{K_P^{-n}(X, Y)\}_{n \in \mathbb{Z}}$. In particular, for a CW pair $(X, Y)$ equipped with a $PU(\mathcal{H})$-bundle $P \to X$, we have the long exact sequence:

$$\cdots \to K_{P|_Y}^{-n-1}(Y) \overset{\delta_{-n-1}}{\to} K_P^{-n}(X, Y) \to K_P^{-n}(X) \to K_{P|_Y}^{-n}(Y) \overset{\delta_{-n}}{\to} \cdots.$$

Note that we can identify $K_P^1(X, Y)$ with $K_P^{-1}(X, Y) = K_{P \times I}(X \times I, Y \times I \cup X \times \partial I)$, and $\delta_0 : K_{P|_Y}^0(Y) \to K_P^1(X, Y)$ with the composition of the following maps:

$$K_{P|_Y}^0(Y) \overset{\beta}{\longrightarrow} K_{P|_Y \times D^2}^0(Y \times D^2, Y \times S^1) = K_{P|_Y}^{-2}(Y) \overset{\delta_{-2}}{\longrightarrow} K_P^{-1}(X, Y).$$

Here $\beta$ induces the Bott periodicity, and is given by "multiplying" a map $T : D^2 \to \mathcal{F}(\mathcal{H})$ representing the generator of $K(D^2, S^1)$.

To construct a similar cohomology theory, we define $KF_P(X, Y)$ by using twisted $\mathbb{Z}_2$-graded vectorial bundles on $X$ whose support do not intersect $Y$. (The *support* of a twisted $\mathbb{Z}_2$-graded vectorial bundle $\mathbb{E} = (\mathcal{U}, (E_\alpha, h_\alpha), \phi_{\alpha\beta})$ on $X$ is the closure of the points $x \in X$ such that $(h_\alpha)_x$ is not invertible for an $\alpha$.) For $n \geq 0$, we put:

$$KF_P^{-n}(X, Y) = KF_{P \times I^n}(X \times I^n, Y \times I^n \cup X \times \partial I^n).$$

Then $K_P^{-n}(X, Y)$ satisfies the (suitably modified) homotopy axiom and the excision axiom in the Eilenberg-Steenrod axioms. In a way parallel to the method in [1], we can also introduce a natural map $\delta_{-n} : KF_{P|_Y}^{-n}(Y) \to KF_P^{-n+1}(X, Y)$, and obtain the long exact sequence for a pair:

$$\cdots \to KF_P^{-1}(X) \to KF_{P|_Y}^{-1}(Y) \stackrel{\delta_{-1}}{\to} KF_P^0(X, Y) \to KF_P^0(X) \to KF_{P|_Y}^0(Y).$$

To extend this sequence, we put $KF_P^1(X, Y) = KF_P^{-1}(X, Y)$ and define $\delta_0 : KF_{P|_Y}^0(Y) \to KF_P^1(X, Y)$ to be the composition of:

$$KF_{P|_Y}^0(Y) \stackrel{\beta}{\longrightarrow} KF_{P|_Y \times D^2}^0(Y \times D^2, Y \times S^1) = KF_{P|_Y}^{-2}(Y) \stackrel{\delta_{-2}}{\longrightarrow} KF_P^{-1}(X, Y),$$

where $\beta$ is given by tensoring a vector bundle representing the generator of $K(D^2, S^1)$. Then the composition of $KF_P^0(X) \to KF_{P|_Y}^0(Y) \to KF_P^1(X, Y)$ is trivial. (This sequence is not yet shown to be exact at this stage.)

The spaces $X \times I^n$ and $Y \times I^n \cup X \times \partial I^n$ in the definition of $KF_P^{-n}(X, Y)$ are used in that of $K_P^{-n}(X, Y)$. Hence the finite-dimensional approximation induces the natural homomorphism $\alpha_{-n} : K_P^{-n}(X, Y) \to KF_P^{-n}(X, Y)$ for $n \geq -1$. We can readily see that $\delta_{-n}$ $(n \geq 1)$ commutes with $\alpha_{-n}$, since $\delta_{-n}$ is essentially defined by an inclusion map of spaces. If $X$ is compact, then $\beta$ commutes with $\alpha_{-n}$, so that $\delta_0$ does. The key to this fact is that the compactness allows us to choose a map $T : D^2 \to \mathcal{F}(\mathcal{H})$ realizing the generator of $K(D^2, S^1)$ in a way appropriate for the finite-dimensional approximation.

Now, for a finite CW complex $X$ and a $PU(\mathcal{H})$-bundle $P \to X$, we can prove the bijectivity of $\alpha_{-n} : K_P^{-n}(X) \to KF_P^{-n}(X)$ $(n \geq 0)$ by the induction on the number of cells in $X$. Notice that, if $P$ is trivial, then an argument by using Proposition 3.4 implies the bijectivity of $\alpha_{-n}$ $(n \geq 0)$. Thus, $\alpha_{-n}$ $(n \geq 0)$ is bijective in the case where $X$ is a point and has only one cell. If the number of the cells in $X$ is $r > 1$, then we can express $X$ as $X = Y \cup e^q$, where $Y \subset X$ is a subcomplex with $(r-1)$ cells, and $e^q \subset X$ is a cell of dimension $q$. We here make use of the commutative diagram $(n \geq 0)$:

$$\begin{array}{ccccccccc}
K_{P|_Y}^{-n-1}(Y) & \to & K_P^{-n}(X, Y) & \to & K_P^{-n}(X) & \to & K_{P|_Y}^{-n}(Y) & \to & K_P^{-n+1}(X, Y) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
KF_{P|_Y}^{-n-1}(Y) & \to & KF_P^{-n}(X, Y) & \to & KF_P^{-n}(X) & \to & KF_{P|_Y}^{-n}(Y) & \to & KF_P^{-n+1}(X, Y).
\end{array}$$

We can assume that the first and the forth columns are bijective in the induction. The excision axiom implies that $K_P^{-n}(X, Y) \cong K_P^{-n}(D^q, S^{q-1})$ and $KF_P^{-n}(X, Y) \cong K_P^{-n}(D^q, S^{q-1})$. Since any $PU(\mathcal{H})$-bundle over $D^q$ is trivial, the second and fifth columns are also bijective. Thus, so is the third column. (The exactness at $KF_{P|_Y}^{-n}(Y)$ is not necessary in the five-term lemma.)

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