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Differential Games with (A)symmetric Players and Heterogeneous Strategies (II)*

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Abstract

One family of heterogeneous strategies in differential games with (a)symmetric players is developed in which one player adopts an *anticipating open-loop strategy* and the other adopts a standard Markovian strategy. Via conjecturing principle, the anticipating open-loop strategic player plans his strategy based on the possible updating the rival player may take. These asymmetric strategies frame non-degenerate Markovian Nash Equilibrium, which can be subgame perfect. Except the stationary path, this kind of strategy makes the study of short-run trajectory possible, which usually are not subgame perfect. However, the short-run non-perfection provides very important policy suggestions.
Keywords: Differential game, subgame perfect Markovian Nash Equilibrium, Heterogeneous strategy, anticipating open-loop strategy.
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1 Introduction

As stated in the seminal paper of Reinganum and Stokey (1985, p.162) "*when formulating a model, care should be taken to choose a strategy space that is appropriate for the situation under study. Path strategies may be appropriate in some situations, decision rule strategies in others, and intermediate formulations in still others.*" More recently, Dockener et al. (2000, p.87) emphasize again that the analysis should " $\cdots$ *consider equilibria in which some of the players represent their optimal control paths in open-loop form while others choose non-degenerate Markovian strategies.* " and further " $\cdots$ *the choice to solve a differential game* $\cdots$ *(for equilibria in which some players use open-loop strategies while others employ non-degenerate Markovian strategies) is part of the modelling stage and one should try to analyze that equilibrium which describes best the situation at hand.*"

This paper takes special care of these intermediate formulations in differential games where all players share the same information structure[1]. Nevertheless, heterogeneity may appear in different aspects.

First, players share the same information set, however, they may have difficulties to perform the same strategies as their rivals. Dawid and Feichtinger (1996) show such asymmetric case in the study of optimal allocation of drug control efforts (see Section 3 for more detail). In their model, both government (the drug controller) and drug dealers know the drug users. But their strategy spaces may differ. Han et al. (2014) present another such kind of situation in dynamic tax competition between large and small economies. As they argued that the small economy may be less efficient than the large one in interpreting its policy and offering public services, while the small economy may be more flexible than its large rival in collective or single-minded action.

Second, though players share the same information and have the same capacities to interpret the same strategies or policies as their rivals, they may differ in their moral standard or their attitudes. In other words, players can be symmetric in their characteristics but just play asymmetric strategies. This can be seen from the international CO_2

[1] See more detail in Section 6.2 of Bacard and Olsder (1998)

emission control problem. Despite the well-known "tragedy of the commons" results, unilateral action has been observed in collective-action problems like climate-change negotiations. Notable in 2007, EU country representatives committed themselves to a unilateral 20 per cent reduction in GHG emissions by 2020 and even offered a 30 per cent decrease in case an international agreement could be found. While the other big players (like the USA and China) can deviate from their commitments (or do not commit at all) and regularly revise their targets and policies. Indeed, under a different setting, Reinganum (1981) shows that under her setting two identical players adopting asymmetric strategies is optimal.

Thus, player engaging in this kind of differential games must first figure out that, among other important issues, in which strategy space he is playing, the subgame perfection of the chosen strategies and the possibility of studying the trajectory dynamics. Via *conjecturing principle*, we shall introduce heterogeneous strategies by allowing some players to adopt open-loop strategies while others non-degenerate Markovian strategies.

The main contribution of this work is that Hamiltonian can be applied to look for heterogeneous strategic Nash Equilibrium along the whole trajectory. The conjecturing principle makes this possible. Otherwise, without this guessing principle, applying Hamiltonian searching for feedback strategies, we will face the difficulties of guessing each others' infinitely many optimal strategies, as clearly stated by Kamien and Schwartz (2003, P.275): " *finding player i's optimal feedback strategy* $u_i^*(x,t)$ *requires that player j's optimal feedback strategy* $u_j^*(x,t)$ *be known which, in turn, requires that player i's optimal be known, and so on.*" The family of games, where one player is less flexible of changing strategies based on state of the world than the other player, makes this guessing process stop at one step instead of infinitely many.

Our approach fits specially well to differential games with unilateral commitment (see more in the Section 3). Most of the economic literature[2] applying differential games ignores the impact of unilateral commitment and focuses only on symmetrical strategy spaces. Nevertheless, unilateral commitment is not the only case where these kinds

[2]See Long (2010) for a recent and complete survey

of heterogeneous strategies are adopted. In Section 3, we present different situations where this kind of strategy space should be taken. Thus far, the literature provides only limited applications of these kinds of heterogeneous strategies. The technique we are going to present is also studied by Dockener et al. (2000; Example 4.1) where there are two asymmetric players with different objective functions though sharing the same state equation.

The paper is organized as follows: in Section 2, we introduce the concept and formulation of heterogeneous strategy in a general setting. Then, Section 3 presents some situations where anticipating open-loop and non-degenerate Markovian strategies should be played at the same time. And as an example of further development and mathematics exercise of calculating the strategy, in Section 4, we restudy the seminal model of Fershtman and Kamien (1987)–*dynamic duopolistic competition with sticky prices*, in which they study the duopolistic competition through time under the assumption that the price of a homogeneous product does not adjust instantaneously to the price indicated by its demand function at the give level of output. By applying the heterogenous strategies developed in Section 2, we can conclude that (i) a different stationary subgame perfect Markovian Nash equilibrium is obtained; (ii) the results of limit game from heterogenous strategies lie between Fershtman and Kamien (1987)'s symmetric open-loop and symmetric Markovian Nash equilibria: in the limit game, the steady state price from heterogenous strategies is strictly higher than the price from symmetric Markovian Nash equilibrium and strictly lower than the price from symmetric open-loop (which is the same as in the static Cournot equilibrium). Some concluding remarks are given in Section 5.

2 Heterogeneous strategies

Consider a two-player differential game. Each player $i(=1,2)$ chooses $u_i \in U_i$ (where $U_i \in \mathbb{R}$ is the choice space for player i) to maximize her objective function Π_i:

$$\max_{u_i} \Pi_i(u_i, u_j) = \max_{u_i} \int_0^\infty e^{-rt} f_i(t, u_i, u_j, x(t))dt, \;\; i,j=1,2, \; i \neq j,$$

where player j's strategy $u_j \in U_j$ is taken as given by player i. We assume $x(t) \in X$ is the shared common state of the system [3] (with $X \in \mathbb{R}$, the state space) and r is a positive constant denoting time preference and is the same for both players for simplicity. The state of the system is given by the following differential equation

$$\dot{x}(t) = g(t, u_1(t), u_2(t), x(t)), \forall t \geq 0 \tag{1}$$

with initial condition $x(0)$ given. For simplicity, we also assume both objective functions $f_i(\cdot)$ and state function $g(\cdot)$ are smooth functions.

For this differential game, we define the heterogeneous strategy as:

Definition 1 ***(Heterogeneous Strategic Nash Equilibrium)*** *A 2-tuple* (Ψ_1, Ψ_2) *of functions* $\Psi_1 : X \times [0, +\infty) \to \mathbb{R}$ *and* $\Psi_2 : [0, +\infty) \to \mathbb{R}$*, with* $\Psi_1 = \Psi_1(x, t), \forall (x, t) \in X \times [0, +\infty)$ *and* $\Psi_2 = \Psi(t), \forall t \in [0, +\infty)$*, is called a Heterogeneous Strategic Nash Equilibrium if, for each* $i = 1, 2$*, an optimal control path* u_i *of player* i *exists and is given by: Markovian strategy for player 1,* $u_1(t) = \Psi_1(x(t), t)$*, and open-loop strategy for player 2,* $u_2(t) = \Psi_2(t)$*.*

In other words, player 1 adopts a Markovian strategy and its optimal choice u_1 depends not only on time t, but also on the current state x of the system; while player 2's optimal strategy u_2 depends only on time, i.e., there is an irrevocable commitment, which is given based on some anticipating of the rival player's potential Markovian choice.

In the following, we build the *principle of conjecturing* on the guessing technique[4]. In short, the equilibrium is framed based on two stages: stage one is an imaginary stage and no game is really played. However, at this stage the players suppose that both of them adopt open-loop strategies based on Pontryagin's Maximum principle, which would provide information for choosing their real strategies in the next stage and conjecturing the other player's strategy. At stage two, the real game starts, players adopt strategies based on their calculated information from stage one.

[3]Except some players commit on all the state variables, the concept and method may not be analogous if there were different states for different players or multiple states, see for example Reynolds (1987).

[4]Guessing technique is often in use in game study, see Long (2010) for more examples.

To be more precise, the process is the following:

Stage 1. Given Pontryagin's Maximum principle can offer optimal solution for each player's optimal control problem provided some sufficient conditions (i.e., concavity of objective functions and state equations) are checked, we can write down player i's Hamiltonian function as

$$\mathcal{H}_i(x, \lambda_i, u_i, t) = f_i(t, u_i(x,t), u_j^*(t), x) + \lambda_i(t) g(t, u_i(x,t), u_j^*(t), x),\ i = 1, 2,\ i \neq j.$$

Here $u_j^*(t)$ is player j's optimal choice and taken as given by player i, λ_i is player i's costate variable.

The usual first order conditions present the optimal solution for each player which also perform the open-loop strategies. Denote these optimal open-loop strategies as: $\Psi_1(t) = \Psi_1(x(t), t)$ and $\Psi_2(t) = \Psi_2(x(t), t)$, $\forall t \geq 0$.

This step is essential for completing this process of heterogeneous strategy.

Stage 2. Player 1 (the Markovian strategic player) takes player 2's (open-loop) strategy $\Psi_2(t)$ as given, and hence, faces the following optimization problem:

$$\begin{cases} \max\limits_{u_1(x,t)} \displaystyle\int_0^{\infty} e^{-rt} f_1(t, u_1(x,t), \Psi_2(t), x) dt, \\ \text{subject to } \ \dot{x}(t) = g(t, u_1(x(t), t), \Psi_2(t), x(t)). \end{cases} \tag{2}$$

The corresponding current-value Hamiltonian for player 1 is

$$\mathcal{H}_1(x, \lambda_1, u_1, t) = f_1(t, u_1(x,t), \Psi_2(t), x) + \lambda_1(t) g(t, u_1(x,t), \Psi_2(t), x),$$

where λ_1 denotes player 1's costate variable.

Player 2, the open-loop strategy player, applies the *conjecturing principle*: player 2 guesses that the strategy of player 1 with the following modification: player 2 guesses that strategy $\Psi_1(t) = \Psi_1(x(t), t)$ will be replaced by $\Psi_1(x, t)$ with *any state variable* $x \in X$, since player 1 plays Markovian strategy. Therefore, player 2's conjecturing of player 1's strategy is: $\Psi_1(x, t)$ for any $(x, t) \in X \times [0, \infty)$.

Remark. The open-loop player differs from the classical case where both players adopt open-loop strategies. Under current setting, the open-loop strategic player corrects, though at the beginning of the game, her strategy based on the possible updating the rival player may take. Thus, the open-loop strategy player is not completely passive, rather in an anticipating defensive position. We call this kind of open-loop strategy as *anticipating open-loop strategy*. This way of constructing strategy gives possibility of subgame perfection.[5]

Thus, player 2 (open-loop strategy player), taking player 1's Markovian strategy $\Psi_1(x,t)$ as given, faces the following problem:

$$\begin{cases} \max\limits_{u_2(t)} \int_0^{\infty} e^{-rt} f_2(t, \Psi_1(x,t), u_2(t), x(t)) dt, \\ \text{subject to } \dot{x}(t) = g(t, \Psi_1(x(t),t), u_2(t), x(t)). \end{cases} \tag{3}$$

Similarly, the corresponding current-value Hamiltonian for player 2 is

$$\mathcal{H}_2(x, \lambda_2, u_2, t) = f_2(t, \Psi_1(x,t), u_2, x) + \lambda_2(t) g(t, \Psi_1(x,t), u_2, x),$$

where λ_2 is the costate variable for player 2.

Given player 2 still plays open-loop strategy, player 1 will conjecture that player 2's strategy is $\Psi_2(t)$– stick to commitments.

The first order condition yields that player 1's choices $u_1(x,t)$ is given by the solution of

$$\frac{\partial \mathcal{H}_1}{\partial u_1} = \frac{\partial f_1}{\partial u_1} + \lambda_1 \frac{\partial g}{\partial u_1} = 0. \tag{4}$$

The costate variable λ_1 verifies equation[6]

$$\dot{\lambda}_1(t) = r\lambda_1 - \frac{d\mathcal{H}_1}{dx} = r\lambda_1 - \left[\left(\frac{\partial f_1}{\partial x} + \lambda_1 \frac{\partial g}{\partial x} \right) + \left(\frac{\partial f_1}{\partial u_1} + \lambda_1 \frac{\partial g}{\partial u_1} \right) \frac{\partial \Psi_1(x,t)}{\partial x} \right],$$

[5]Of course, open-loop strategy may be subgame perfect, see Reynold(1987).

[6]A similar notation can also be found in Itaya and Shimomura (2001). However, without guessing process, they can not go further than just writing down the functional form of first order conditions. See more systematic statement for the general case in Kamien and Schwartz (2003, Page 275).

where $\frac{d\mathcal{H}_1}{dx}$ denotes the **total derivative**[7] of $\mathcal{H}_1$ with respect to x and the last term is equal to zero, by the first order condition (4). Thus, player 1's co-state equation reads

$$\dot{\lambda}_1(t) = r\lambda_1(t) - \left(\frac{\partial f_1}{\partial x} + \lambda_1(t)\frac{\partial g}{\partial x}\right) \tag{5}$$

with transversality condition $\lim_{t\to\infty} e^{-rt}\lambda_1(t)x(t) = 0$.

Similarly, $u_2(t)$, the optimal choices of player 2, is given by

$$\frac{\partial \mathcal{H}_2}{\partial u_2} = \frac{\partial f_2}{\partial u_2} + \lambda_2 \frac{\partial g}{\partial u_2} = 0. \tag{6}$$

And, the costate equation is

$$\dot{\lambda}_2(t) = r\lambda_2 - \frac{d\mathcal{H}_2}{dx} = r\lambda_2 - \left[\left(\frac{\partial f_2}{\partial x} + \frac{\partial f_2}{\partial u_1}\frac{\partial \Psi_1(x,t)}{\partial x}\right) + \lambda_2\left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial u_1}\frac{\partial \Psi_1(x,t)}{\partial x}\right)\right]$$

or equivalently,

$$\dot{\lambda}_2(t) = r\lambda_2 - \left[\left(\frac{\partial f_2}{\partial x} + \lambda_2\frac{\partial g}{\partial x}\right) + \left(\frac{\partial f_2}{\partial u_1} + \lambda_2\frac{\partial g}{\partial u_1}\right)\frac{\partial \Psi_1(x,t)}{\partial x}\right]. \tag{7}$$

The associated transversality condition is $\lim_{t\to\infty} e^{-rt}\lambda_2(t)x(t) = 0$.

Remark. Equation (7) differs from (5) due to the fact that in (5) the first order condition (4) can be applied; while in (7), that is not possible.

Denote the solution of (4) and (6) as $u_1^* = \Psi_1^*(x,t), \forall (x,t) \in X \times [0,\infty)$ and $u_2^* = \Psi_2^*(t), \forall t \geq 0$, respectively. To be more precise $\Psi_1^*(x,t)$ is a function of state x, the costate variable evaluated at time t; and $\Psi_2^*(t)$ is a function of state and costate variables both evaluated at t, thus, $\Psi_1^*(x,t) = \Psi_1^*(x,\lambda_1(t),t)$ and $\Psi_2^*(t) = \Psi_2^*(x(t),\lambda_2(t),t)$. Substituting these two into the Hamiltonian, we can readily check that the maximized Hamiltonian $\mathcal{H}_1^*(x,\lambda_1,t)$ and $\mathcal{H}_2^*(x,\lambda_2,t)$ are given by

$$\mathcal{H}_1^*(x,\lambda_1,t) = f_1(t,\Psi_1^*(x,\lambda_1,t),\Psi_2^*(x,\lambda_2,t),x) + \lambda_1\, g(t,\Psi_1^*(x,\lambda_1,t),\Psi_2^*(x,\lambda_2,t),x)$$

[7] The state variable affects the current-value Hamiltonian via two different ways: the direct impacts by the state equation and indirect impacts due to the strategy of the other player.

and

$$\mathcal{H}_2^*(x, \lambda_2, t) = f_2(t, \Psi_1^*(x, \lambda_1, t), \Psi_2^*(x, \lambda_2, t), x) + \lambda_2\, g(t, \Psi_1^*(x, \lambda_1, t), \Psi_2^*(x, \lambda_2, t), x).$$

If sufficient concavity and smoothness conditions on the objective functions $f_i(t, u_1, u_2, x)$ and state function $g(t, u_1, u_2, x)$ can be imposed, the maximized Hamiltonian are concave with respect to the state variable x. Then, by[8] Theorem 3.2 (Dockner et al, 2000), $u_i^*(t)$ $(i = 1, 2)$ are optimal paths. Thus, the solution $\{u_1^*(x, t), u_2^*(t)\}$, $\forall x \in X$ and $\forall t \geq 0$ form a pair of non-degenerate Markovian Nash Equilibrium.

Finally, substituting $u_1^* = \Psi_1^*(x, t)$ and $u_2^* = \Psi_2^*(t)$ into the canonical system: state equation (1), two costate equations (5) and (7), we can obtain the solution for the whole trajectory path of the differential game.

We close this section with a brief remark concerning the subgame perfection of stationary Markovian Nash Equilibrium in an autonomous system. The pair of stationary strategy, $(\overline{u_1}^* = \Psi_1^*(\overline{x}), \overline{u_2}^* = \Psi_2^*(\overline{x}))$ is subgame perfect as long as it does not depend on the initial condition $x(0)$. The reason for subgame perfection follows the remark of Dockner et al (2000, P.105)– if an autonomous differential game is defined on the time interval $[0, \infty)$, then its subgame is equivalent (in fact, identical) to the original game. It follows from the definition of subgame perfect Nash equilibrium that any stationary Markovian Nash Equilibrium is subgame perfect, provided it is independent of the initial state $x(0)$. While if the system is non-autonomous, this results may not be true, even the stationary strategy is independent of the initial state.

3 Examples of games with heterogeneous strategies

In this section, we provide some examples where asymmetric situations appear and heterogenous strategy spaces should be adopted by the players. The order of examples are the following: Example 1 and 2 present the situation where the two players

[8]It is easy to imagine that in quite some games this sufficient condition failed to hold. In Section 4, we face one of these cases where we have to use other methods to check the sufficiency.

enjoy the same information set, however, one does not have the same capacity of taking the same strategy as her rival. Example 3 is the case where the two players can be symmetric or identical in obtaining information of the state of the world and having the same capacity of making decision, however, due to social choices, political considerations or other constraints, one player commits to and keeps a strategy, while the other player updates her strategy based on the state of the world. This is the case we called *unilateral commitment* in differential game.

This list of real world situations in which dynamic heterogeneous strategies are played is not exhaustive. Further applications and potential examples and exercises will be mentioned again in the conclusion.

3.1 Same information set but different strategy spaces

Example 1. Optimal allocation of drug control efforts

Dawid and Feichtinger (1996) provide a dynamic drug control problem with two players, i.e., drug dealers and the government. They offer the optimal allocation of governmental efforts between treatment and law enforcement minimizing the total discounted cost stream in the equilibrium. Their players are asymmetric and their game is not linear-quadratic.

More precisely, their model is the following: both government and drug dealers maximize their respective objectives. The drug dealers choose, $u > 0$, the effort of the dealers, which they interpret as the time the dealer spends in the street in order to attract new customers; and government chooses, $v > 0$, the whole expense spent to deal with drug problem. Denote $x(t) \in [0, \overline{x}]$ as drug user at time t, with given fixed upper bound $\overline{x} > 0$.

Dawid and Feichtinger (1996) assume that the growth of the stock of drug users is governed by three forces: activities of the drug dealers, death and treatment of drug users. Denote $g(x)$ as a growth function with the characteristic notions of a diffusion dynamics of drug user, d as death rate and hence, the motion of drug user follows:

$$\begin{cases} \dot{x} = g(x)\sqrt{u} - dx - f(x)\sqrt{\phi v}, \\ x(0) = x_0 \in [0, \overline{x}], \quad \phi \in [0,1], d > 0. \end{cases} \tag{8}$$

Here, function $f(x)\sqrt{\phi v}$ measures the treatment effects, with $\phi \in [0,1]$ fraction of budget for drug control invested in treatment, $(1-\phi)v$ the fraction of budget of crackdown on dealers, and $f \in C^1[0, \overline{x}]$ checks: $f(0) = 0$, $f'(x) > 0$ and $(f(x)/x)' < 0$.

The objective of drug dealer is

$$\max_{u \in [0,\infty)} J_d = \int_0^\infty e^{-r_d t}[U(x) - C_d(u, (1-\phi)v)]dt, \tag{9}$$

subject to constraint (8), where function $U(\cdot)$ is dealer's income and function $C_d(\cdot)$ describes the damage for the whole class of dealers caused by government law enforcement actions.

The problem of government is the cost produced by the drug users, including the direct cost, $D(x)$, and the effort of controlling the drug problem, $C_g(v)$. Thus, the objective of government is:

$$\max_{v \in [0,\infty)} J_g = \int_0^\infty e^{-r_g t}[-D(x) - C_g(v)]dt, \tag{10}$$

subject to constraint (8).

Dawid and Feichtinger (1996) develop an explicit solution of the stationary feedback strategies where both players play Markovian strategies. Arguably, though both government and drug dealers have the same information as to the number of drug users, they have different objective and different constraint on taking their strategic actions. Thus, different strategic spaces may be more proper: government's treatment and law enforcement are based on law and/or regulations while the drug dealers act in ways to avoid government control. Given the law and regulations are transparent and announced by the government(if we do not consider policemen's action) while drug dealers do not communicate on their actions and strategies, heterogeneous strategies could be applied in such a context: the drug dealers change their strategies and efforts based on the reality–Markovian strategy, while government action is more open-loop by taking into account the drug dealers' effort, thus anticipating open-loop strategy. Therefore, the two players reach to one heterogeneous strategic Nash Equilibrium.

Example 2. Dynamic tax competition between unequal size jurisdictions

A second illustration stems from the dynamic competition between big economies and small ones such as tax/infrastructre competition among different jurisdictions. Most models in the tax competition literature are static. Though Zissimos and Wooders (2008) have called for the need of dynamic studies of tax competition, Han et al. (2014) is one of the very few exceptions studying tax competition under a dynamic-strategic setting. They assume that a small economy and a large one enter a dynamic tax-and-infrastructure-competition game where the size asymmetry of these two economies play an essential role.

As they argued, on the one hand, small-in-size is a natural disadvantage for most of the small economies; but, on the other hand, small-in-size sometimes can be considered an asset (Kuznets, 1960; Easterly and Kray, 2000) given the economic success of many micro-states. Han et al (2014) argue that small states are more flexible in their political decision making than much larger countries (see also Streeten, 1993); for example, problems related to collective action can be solved more easily in small countries. These attributes facilitate greater single-mindedness and focus on economic policy-making and promote a more rapid and effective response to exogenous change (Armstrong and Read, 1995).

Han et al. (2014) introduce dynamic firm relocation process via location attachment assumption. More precisely, they assume that the one person one capital firm produces net $q+a_i$ units final goods and sold in world competitive market with price normalized to 1. Here, q presents private firm's productivity and a_i, $i = 1, 2$, reads country i's specific productivity enhancing public goods.

Consider an entrepreneur in the small country, if she invests at home, the profit is

$$\pi_1(t) = q(t) + a_1(t) - T_1(t)$$

and if she invests abroad, profit is then

$$\pi_2(t) = q(t) + a_2(t) - T_2(t) - k \cdot x(t)$$

with $T_i(t)$ tax rate in country $i = 1, 2$. Here, mobility cost is $k \cdot x(t)$ with unit mobility cost k and distance to frontier x.

Similar arguments are also true for firms originally located in big country. Thus, we can obtain the indifferent firm is located at

$$x(t, a_1, a_2, T_1, T_2) = \frac{a_2(t) - T_2(t)}{k} - \frac{a_1(t) - T_1(t)}{k}.$$

If denote number of firms at time t located in the small economy as $S_1(t) = S(t)$ and the number of firms in the large economy is then $S_2(t) = 1 - S(t)$ by assuming that total number of firms is constant and normalized to 1. Thus, law of motion of number of firm in the small economy is governed by

$$\dot{S}(t) = -x = \frac{a_1(t) - T_1(t)}{k} - \frac{a_2(t) - T_2(t)}{k}. \tag{11}$$

Furthermore, both small and large economies' policy makers choose tax $T_i(t)$ and public goods $a_i(t)$ simultaneously to maximize tax revenue:

$$J_i = \max_{a_i, T_i} \int_0^{+\infty} e^{-rt} \left(\sqrt{S_i T_i} - \frac{\beta_i}{2} a_i^2 \right) dt, \quad i = 1, 2,$$

subject to dynamic relocation of firms (11).

Han et al. (2014) take into account the differences between large and small economies. And hence, they consider a situation where the small economy plays Markovian strategy while the large country adopts anticipating open-loop strategy. They find that the extra flexibility in policy making– taking Markovian strategy – is very essential for the surviving of small states while competing with big folks.

Another this kind of example could come from a recent work of Shimomura and Thisse (2012). In a static setting, they study asymmetric competition among big and small firms, in which the few big commercial or manufacturing firms are able to affect the market outcome, and a myriad of small family-run businesses with very few employees has a negligible impact on the market. In a general-equilibrium setting, they demonstrate that (abstract on Page 1) "*due to the higher toughness of the market, the entry of big firms leads them to sell more through a market expansion effect, which is generated by the exit of small firms*." Thus, asymmetric strategies are played at the same time depending on the market power. It would be interesting to study the dynamics of this situation and its long-run outcome under a differential game where the market share (such as in Han et al. 2014) or the goods' prices (such as in Fershtman and Kamien, 1987, 1990) could serve as the state variable.

3.2 Symmetric players but heterogenous strategies

Example 3. Transboundary pollution– Unilateral commitments

This example relates to the international CO_2 emission control problem. Since the seminal paper of Dockner and Sorger (1996), there have been various contributions using differential games to study transboundary pollution control problems. However, most of them have simply ignored open-loop strategies. The main reason is that it was thought that players are too naive to play open-loop strategies as they do not use any information acquired during the game and, consequently, do not respond to changes in the current pollution stock. In the context of climate change and policies that have been implemented to its mitigation, it is important to distinguish between countries which have taken binding commitments to stabilize/reduce their Greenhouse Gas (GHG) emissions, such as in the Kyoto Protocol or the EU 2007 Energy Package, and countries which have proposed more flexible approaches, based on regular updates of the targets to reach, according to current states of the world.

For this kind of problem, the EU, under the Kyoto Protocol or its own unilateral policies, makes commitments about emission reductions from which they cannot deviate, while the USA and China, on the contrary, can deviate from their commitments (or do not have any commitment at all) and regularly revise their targets and policies. In the first case, we refer to open-loop strategies and in the second case to Markovian strategies. And, obviously, the classical method, where either both players adopt open-loop strategies or both play Markovian strategies, is not the proper one in these sorts of circumstances.

In this type of example, the choices of different players could be the abatement efforts or CO_2 reduction and the common state will be the environmental quality or CO_2 stock. More precisely, the possible model is the following.

These two players undergo the same pollution state, $x(t)$, which is given by the following equation

$$\dot{x}(t) = E(t) - (u_i + u_j)\sqrt{x(t)} - \delta x(t), \;\; t \geq 0, \tag{12}$$

where the initial condition $x(0)$ is a given positive constant, and parameter $\delta \in [0, 1]$ measures the pollution absorption rate of nature. $E(t) = E_i(t) + E_j(t)$ is a known positive function of pollution emissions.

The two countries' policy makers need to choose their abatement rate u_l, $l = i, j$, to maximize their utility

$$\max_{u_l} \int_0^{\overline{T}} e^{-r_l t} \left(-x(t) - \frac{\alpha_l}{2} u_l^2 \right) dt + S(x(\overline{T})), \;\; l = i, j, \tag{13}$$

subject to the state constraint (12), where $r_l \in [0, 1)$ is the time preference parameter, α_l is a positive, constant adjustment cost coefficient. We consider $\overline{T} \leq \infty$, where $S(x(\infty)) = 0$ and in finite time $\overline{T} = T$, $S(x(T))$ is a given known positive function, with $S_x < 0$. Furthermore, $x(\overline{T})$ is the final target of the pollution state of the world, where both players agree on a final date T.

Most probably, as long as $T < \infty$, the heterogenous strategy is not subgame perfect, which coincides with the tragedy-of-the-commons' outcome, this unilateral decision should have worsened the welfare of EU citizens.

4 The revisit of Fershtman and Kamien's (1987) model "Dynamic Duopolistic Competition with Sticky Prices"

Fershtman and Kamien (1987, 1990) study the *Dynamic duopolistic competition with sticky prices* with infinite-horizon and finite-horizon of time, respectively. The main objective of the first paper is to investigate the relationship between the speed at which the price converges to its value on the static demand function and the resultant stationary subgame perfect Markovian equilibrium price. The second paper specially studies the relationship between the "turnpike properties" of the finite-horizon subgame perfect equilibrium strategies and the infinite-horizon subgame perfect equilibrium strategies where the feedback strategies in a finite-horizon game are non-autonomous.

In the following, we first restate the linear-quadratic model of Fershtman and Kamien (1987) and repeat their findings which will be used to compare with the outcomes of heterogeneous strategy. The following subsections are devoted to the process of finding the heterogeneous-strategic subgame perfect Nash Equilibrium.

Remark. Before starting, we must mention that this section is more a suggestive mathematics' exercise of how to employ the heterogeneous strategy in a differential game rather than looking for new insight from their classical model.

Without economic interpretation, the mathematics model of Fershtman and Kamien (1987) is the following: let $i = 1, 2$. Player i chooses u_i to maximize objective

$$\max_{u_i} J^i(u_i, u_j) = \int_0^\infty e^{-rt} \left(pu_i(t) - cu_i(t) - \frac{u_i^2(t)}{2} \right) dt$$

subject to

$$\dot{p}(t) = s(a - (u_1(t) + u_2(t)) - p(t))$$

and $p(0) = p_0$ given, with positive constants r, a and c given.

4.1 Main results of Fershtman and Kamien (1987)

Fershtman and Kamien (1987) demonstrate the following results (see also Fershtman and Kamien, 1990, page 51-52).

(A) (Open-loop Strategy) The standard current value Hamiltonian yields the following necessary conditions for an open-loop equilibrium:

$$p(t) - c - u_i(t) - \lambda_i(t)s = 0, \quad i = 1, 2,$$

$$\dot{\lambda}_i(t) = (s + r)\lambda_i(t) - u_i(t), \text{ and } \lim_{t\to\infty} e^{-rt}\lambda_i(t) = 0, \;\; i = 1, 2.$$

(B) (Markovian Strategy) The following strategies constitute an asymptotically-stable feedback Nash equilibrium for the infinite-horizon game

$$u_i^*(p) = \begin{cases} 0, & p \le \widehat{p} \\ (1 - sK_\infty)p + (sE_\infty), & p > \widehat{p}, \;\; i = 1, 2 \end{cases} \tag{14}$$

where

$$K_\infty = (r + 6s - \sqrt{(r + 6s)^2 - 12s^2})/6s^2,$$

$$E_\infty = \frac{-asK_\infty + c - 2scK_\infty}{r - 3s^2K_\infty + 3s}$$

and

$$\widehat{p} = \frac{c - sE_\infty}{1 - sK_\infty}.$$

(C) (Limit Game) As the speed of adjustment goes to infinity, the static Cournot equilibrium price, $p_D = \frac{a+c}{2}$, is the asymptotic limit of the open-loop Nash equilibrium, $p^*_{open} = \dfrac{4sp^*_D + 3rp^*_c}{4s + 3r}$ (with $p^*_c = \frac{a+2c}{3}$), which is not subgame perfect, while the globally stable symmetric Markovian Nash equilibrium price, $p^*_{MSPN} = \dfrac{p^*_c + 2\sqrt{2/3}p^*_D}{1 + 2\sqrt{2/3}}$, converges to a value below it.

4.2 Non-degenerate Markovian Nash Equilibrium- Heterogeneous strategies

In this subsection, we present the heterogenous strategies introduced in the previous section and show that the equilibrium is indeed one non-degenerate Markovian Nash equilibrium.

Suppose player 1 plays open-loop strategy, $u_1(t)$, by guessing player 2's Markovian strategy. Player 2 plays Markovian strategy, $u_2(p,t), \forall p > 0$, by knowing that player 1's open-loop strategy. Then, player 1's problem is

$$\max_{u_1(t)} J^1(u_1, u_2) = \int_0^\infty e^{-rt} \left(pu_1(t) - cu_1(t) - \frac{u_1^2(t)}{2} \right) dt$$

subject to

$$\dot{p}(t) = s(a - (u_1(t) + u_2^*(p(t), t)) - p(t)).$$

Player 2's problem is

$$\max_{u_2(p,t)} J^2(u_1, u_2) = \int_0^\infty e^{-rt} \left(pu_2(t) - cu_2(t) - \frac{u_2^2(p,t)}{2} \right) dt$$

subject to

$$\dot{p}(t) = s(a - (u_1^*(t) + u_2(p(t), t)) - p(t)).$$

Thus, player 1's Hamiltonian is

$$\mathcal{H}^1(p, u_1, \mu_1, t; u_2^*(p,t)) = \left(pu_1(t) - cu_1(t) - \frac{u_1^2(t)}{2} \right) + \mu_1 s(a - (u_1(t) + u_2^*(p,t)) - p)$$

where $u_2^*(p(t), t)$ is player 2's optimal strategy and μ_1 is player 1's costate variable.

The first order condition yields (necessary and sufficient, due to strict concavity of Hamiltonian in term of control variable),

$$u_1^*(t) = p(t) - c - s\mu_1(t)$$

and

$$\dot{\mu_1}(t) = r\mu_1(t) - \frac{\partial \mathcal{H}^1}{\partial p} = r\mu_1(t) - \left[u_1 - s\mu_1 - s\mu_1 \frac{\partial u_2^*(p,t)}{\partial p} \right].$$

Following the conjecture principle presented in the previous section, player 1 **guesses** that player 2's strategy is given by:

$$u_2^*(p,t) = p - c - s\mu_2(t)$$

with $\mu_2(t)$ costate variable of player 2.

Thus, player 1's costate equation is

$$\dot{\mu_1}(t)) = (r+2s)\mu_1(t) - u_1(t),$$

with trasversality condition $\lim_{t\to\infty} e^{-rt}\mu_1(t)p(t) = 0$.

Similarly, player 2's Hamiltonian is

$$\mathcal{H}^2(p, u_2, \mu_2, t; u_1^*(t)) = \left(pu_2(t) - cu_2(t) - \frac{u_2^2(t)}{2}\right) + \mu_2 s(a - (u_1(t) + u_2^*(p,t)) - p),$$

where $u_1^*(t)$ is player 1's optimal strategy. And the first order condition gives (necessary and sufficient, due to strict concavity of Hamiltonian in term of control variable),

$$u_2^*(p,t) = p - c - s\mu_2(t),$$

and

$$\dot{\mu_2}(t) = r\mu_2(t) - \frac{\partial \mathcal{H}^2}{\partial p} = (r+s)\mu_2(t) - u_2(t),$$

with trasversality condition $\lim_{t\to\infty} e^{-rt}\mu_2(t)p(t) = 0$.

It is easy to check that both maximized Hamiltonian of player 1 and 2 are convex in term of state variable p. Therefore, we can not use directly the sufficient condition of Theorem 3.2 or Theorem 4.2 of Dockner et al. (2000) to get the sufficiency and can not state what we obtained are indeed Markovian strategies. Thus, we shall use the basic definition of optimality: we need to prove that for player 1,

$$J^1(u_1^*, u_2^*) \geq J^1(u_1, u_2^*),\ \forall u_1 \geq 0$$

and for player 2,

$$J^2(u_1^*, u_2^*) \geq J^1(u_1^*, u_2),\ \forall u_2 \geq 0.$$

By definition,

$$
\begin{aligned}
&J^1(u_1^*, u_2^*) - J^1(u_1, u_2^*) \\
&= \int_0^\infty e^{-rt} \left[\left(pu_1^*(t) - cu_1^*(t) - \frac{u_1^{*2}(t)}{2} \right) - \left(pu_1(t) - cu_1(t) - \frac{u_1^2(t)}{2} \right) \right] dt \\
&= \int_0^\infty e^{-rt} \left[(\mathcal{H}^1(p^*, u_1^*, \mu_1; u_2^*(p^*)) - \mu_1 \dot{p}^*) - (\mathcal{H}^1(p, u_1, \mu_1; u_2^*(p)) - \mu_1 \dot{p}) \right] dt \\
&\geq \int_0^\infty e^{-rt} \left[(\mathcal{H}^1(p^*, u_1^*, \mu_1; u_2^*(p^*)) - \mathcal{H}^1(p, u_1^*, \mu_1; u_2^*(p))) - \mu_1 \cdot (\dot{p}^* - \dot{p}) \right] dt
\end{aligned}
$$

where the last inequality comes from the fact that $\mathcal{H}^1$ is strictly concave in u_1, that is,

$$
\mathcal{H}^1(p, u_1, \mu_1; u_2^*(p)) \leq \mathcal{H}^1(p^*, u_1^*, \mu_1; u_2^*(p^*))
$$

and hence

$$
-\mathcal{H}^1(p, u_1, \mu_1; u_2^*(p)) \geq -\mathcal{H}^1(p^*, u_1^*, \mu_1; u_2^*(p^*)).
$$

By mean value theorem, we have that there exists ξ lies between p and p^*, such that

$$
\begin{aligned}
&\mathcal{H}^1(p^*, u_1^*, \mu_1; u_2^*(p^*)) - \mathcal{H}^1(p, u_1^*, \mu_1; u_2^*(p)) = \tfrac{d\mathcal{H}^1}{dp}(\xi, u_1^*, \mu_1; u_2^*(\xi)) \cdot (p^* - p) \\
&= \left[u_1^* + s\mu_1 \left(-1 - \frac{\partial u_2^*(\xi)}{\partial p} \right) \right] \cdot (p^* - p) \\
&= (u_1^* - 2s\mu_1) \cdot (p^* - p).
\end{aligned}
$$

Therefore,

$$
J^1(u_1^*, u_2^*) - J^1(u_1, u_2^*) \geq \int_0^\infty e^{-rt} (u_1^* - 2s\mu_1) \cdot (p^* - p) dt - \int_0^\infty e^{-rt} \mu_1 \cdot (\dot{p}^* - \dot{p}) dt.
$$

Integration by parts of the second part, it yields

$$
\begin{aligned}
&\int_0^\infty e^{-rt} \mu_1 (\dot{p}^* - \dot{p}) dt \\
&= e^{-rt} \mu_1 (p^*(t) - p(t)) \mid_0^\infty - \int_0^\infty (p^* - p) \frac{d}{dt} (e^{-rt} s\mu_1(t)) dt \\
&= 0 - \mu_1(0)(p^*(0) - p(0)) - \int_0^\infty (p^* - p) e^{-rt} (-r\mu_1 + \dot{\mu_1}) dt \\
&= -\int_0^\infty (p^* - p) e^{-rt} (-r\mu_1 + \dot{\mu_1}) dt,
\end{aligned}
$$

by the fact $p^*(0) = p(0) = p_0$ and the transversality conditions.

Thus,

$$J^1(u_1^*, u_2^*) - J^1(u_1, u_2^*) \geq \int_0^\infty e^{-rt} \left[u_1^* - 2s\mu_1 - r\mu_1 + \dot{\mu_1} \right] dt = 0$$

by the first order condition.

Hence, for all $u_1 > 0$, we have

$$J^1(u_1^*, u_2^*) \geq J^1(u_1, u_2^*).$$

In other words, $u_1^*(t) = p(t) - c - s\mu_1(t)$, indeed, forms an open-loop strategy for player 1.

Similarly, we can prove that for player 2

$$J^2(u_1^*, u_2^*) \geq J^2(u_1^*, u_2), \ \forall u_2 > 0.$$

Thus, $u_2^*(p, t) = p - c - s\mu_2(t)$ is one Markovian strategy for player 2.

And therefore, the pair (u_1^*, u_2^*) forms one non-degenerate Markovian Nash equilibrium by definition.

4.3 Stationary Subgame Perfect Markovian Nash Equilibrium

Rewrite the canonical system as following

$$\begin{cases} \dot{p}(t) = s(a - (u_1 + u_2) - p(t)), \\ \dot{\mu_1}(t)) = (r + 2s)\mu_1(t) - u_1(t), \\ \dot{\mu_2}(t) = (r + s)\mu_2(t) - u_2(t), \end{cases}$$

with

$$u_1(t) = p(t) - c - s\mu_1(t), \ \ u_2(p, t) = p - c - s\mu_2(t).$$

Substituting u_1, u_2 into the dynamic equation, we have

$$\begin{cases} \dot{p}(t) = s(a + 2c - 3p + s(\mu_1 + \mu_2)), \\ \dot{\mu_1}(t)) = (r + 3s)\mu_1(t) - p(t) + c, \\ \dot{\mu_2}(t) = (r + 2s)\mu_2(t) - p(t) + c. \end{cases} \tag{15}$$

Hence, the Jacobian matrix of system (15) is

$$J(p, \mu_1, \mu_2) = \begin{pmatrix} -3s & s^2 & s^2 \\ -1 & r + 3s & 0 \\ -1 & 0 & r + 2s \end{pmatrix}, \tag{16}$$

which has three eigenvalues: ξ_i, $i = 1, 2, 3$. It is straightforward to see that $\sum_{i=1}^{3} \xi_i = trace(J) = 2r + 2s > 0$ and $\prod_{i=1}^{3} \xi_i = \det(J) = -3sr^2 - 13rs^2 - 13s^3 < 0$. Positive trace states that there are positive eigenvalues or at least positive real parts if complex eigenvalues appear, and the negative determinant reads that there is at least one negative eigenvalue. Combining these two parts together, we can claim that there is one and only one negative eigenvalue which we denote as $\xi_1(< 0)$.

The steady state of system (15) is

$$\overline{\mu_1} = \frac{\overline{p} - c}{r + 3s}, \quad \overline{\mu_2} = \frac{\overline{p} - c}{r + 2s}$$

and

$$\overline{p} = \frac{a + 2c - sc\left(\frac{1}{r+2s} + \frac{1}{r+3s}\right)}{3 - s\left(\frac{1}{r+2s} + \frac{1}{r+3s}\right)} \tag{17}$$

which is independent of the initial condition p_0. Furthermore, the unique globally asymptotically stable path can be give by

$$p(t) = \overline{p} + (p_0 - \overline{p})e^{\xi_1 t}, \tag{18}$$

and the convergence to its long-run equilibrium is also independent of the initial condition p_0. Thus, the corresponding stationary strategies

$$\overline{u_1^*}(\overline{p}) = \overline{p} - c - s\overline{\mu_1} = \frac{r + 2s}{r + 3s}(\overline{p} - c), \quad \overline{u_2^*}(\overline{p}) = \overline{p} - c - s\overline{\mu_2} = \frac{r + s}{r + 2s}(\overline{p} - c)$$

are independent of the initial state p_0 as well.

Following the similar notation as in Fershtman and Kamien (1987), we conclude the above analysis as a theorem.

Theorem 1 *Suppose $p > c$. Let $u_1^*(p) = \frac{r+2s}{r+3s}(p-c)$ and $u_2^*(p) = \frac{r+s}{r+2s}(p-c)$. Then, for the infinite horizon autonomous differential game under consideration,*

(b) there is a stationary heterogenous Markovian Nash equilibrium price given by (17) and its unique globally asymptotically stable path given by (18) with ξ the negative eigenvalue of matrix (16).

(a) (u_1^*, u_2^*) constitutes a heterogeneous global asymptotic stable Markovian subgame perfect Nash Equilibrium;

Economic interpretation: Player 1 adopts an *anticipating* open-loop strategy which is based on guessing the Markovian strategy of player 2. If player 2 plays subgame perfect stationary strategy, player 1, as a by-product, also plays subgame perfect strategy. The reason is that the guessing process includes all the information about what player 2's optimal strategy will be. That differ from the case where both players adopt open-loop strategies. Thus, the updating information is embodied in the guessing process rather than losing in commitments.

4.4 The limit game and comparison

Fershtman and Kamien (1987) compare the outcomes of different strategies under the limit game when $s \to +\infty$ or $r \to 0$. We rewrite their notation here: static Cournot equilibrium $p_D^* = \dfrac{a+c}{2}$, symmetric open-loop stationary equilibrium $p_{open}^* = \dfrac{4sp_D^* + 3rp_c^*}{4s+3r}$ and symmetric Markovian subgame perfect Nash equilibrium $p_{MSPN}^* = \dfrac{p_c^* + 2\sqrt{2/3}p_D^*}{1+2\sqrt{2/3}}$ with $p_c^* = \frac{a+2c}{3}$.

In the limit game, Fershtman and Kamien (1987) demonstrate that

$$\lim_{s\to+\infty} p^*_{open} = p^*_D > \lim_{s\to+\infty} p^*_{MSPN}.$$

Similarly, taking limit in (17) yields

$$\lim_{s\to+\infty} \overline{p} = \frac{6a+7c}{13}.$$

It is easy to check that

$$\lim_{s\to+\infty} p^*_{open} - \lim_{s\to+\infty} \overline{p} = \frac{a-c}{26} > 0$$

and

$$\lim_{s\to+\infty} \overline{p} - \lim_{s\to+\infty} p^*_{MSPN} = \frac{(5-\sqrt{6})(a-c)}{13(3+2\sqrt{6})} > 0,$$

where $a > c$ is a basic assumption in the original paper of Fershtman and Kamien (1987, see page 1155, equ (2.2) in Theorem 1).

The above analysis is concluded in the following.

Proposition 1 *In the limit game, the stationary heterogenous strategic Markovian Nash equilibrium price lies strictly between the prices of limit symmetric open-loop Nash equilibrium and the limit symmetric Markovian Nash equilibrium.*

5 Concluding remarks

Some particular heterogeneous strategies are introduced in various kinds of differential games. The heterogeneity means that one player adopts an anticipating open-loop strategy while the other player adopts a non-degenerate Markovian strategy - thus the strategy spaces are heterogeneous. The key idea is the guessing of the rival's strategies and the construction of strategy could be specially useful to the study of asymmetric players' differential games.

The novelty of this kind of strategy is twofold. On the one hand, it offers another – except the standard Markovian strategy for all players – stationary subgame perfect non-degenerate Markovian Nash equilibrium for an autonomous system with infinite-horizon. On the other hand, via Hamiltonian, it can characterize the whole trajectory, which is especially useful in the case of asymmetric players' non-linear-quadratic differential games. However, it may be hard to prove its subgame perfection. As Bertinelli et al. (2014) notice, the short-run trajectory may be very different from the long-run stationary solution.

Nevertheless, the strategy's finite-horizon disadvantage itself offers useful information to some differential games where unilateral commitments happen. For instance, in the case of environmental problem we mentioned in Section 3, a short-run committed policy is most likely not subgame perfect and hence, introducing this kind of policy will not only increase the free riding problem, but also hurt the player's own welfare. A similar situation can also happen if there were a plan of shutting down tax havens in finite-time, such as in the OECD's Harmful Tax Practices Initiative (see, for example, Elsayyad and Konrad, 2012). In this game, the big player, here the OECD, usually is against tax havens and plays anticipating open-loop strategy while the tax havens play Markovian strategy. Due to the non-subgame perfection, we may conclude again, as Elsayyad and Konrad (2012) " that reducing the number of tax havens but not eliminating them altogether as a 'big bang', may reduce welfare in the OECD."

Therefore, for future research, the special care should be fourfold. (1) The study of short-run loss due to unilateral commitments (or actions) compared with those where all players adopt similar strategies and feedback information is fully exploited. Here the loss could include welfare only or welfare plus some externality. (2) Attention may be given to the case where multiple states arise, such as Reynolds (1987). It may be the case that even committed players only commit on some state variables rather than on all. (3) In case of multiple instruments' competition, some of the control variables depend on some states and time while the other control variables depend on time only. And that is differ from our setting here as well. (4) Currently, we take into account the heterogeneous characters of players who share the same information structure, further development should be given where players sharing different dynamic information, then what would be the correct or proper strategy spaces?

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