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Abstract

This paper provides a thorough analysis of oligopolistic markets with positive demand-side network externalities and perfect compatibility. The minimal structure imposed on the model primitives is such that industry output increases in a firm's rivals' total output as well as in the expected network size. This leads to a generalized equilibrium existence treatment that includes guarantees for a nontrivial equilibrium, and some insight into possible multiplicity of equilibria.

We formalize the concept of industry viability and show that it is always enhanced by having more firms in the market and/or by technological improvements. We also characterize the effects of market structure on industry performance, with an emphasis on departures from standard markets. The approach relies on lattice-theoretic methods, which allow for a unified treatment of various general results in the literature on network goods. Several illustrative examples with closed-form solutions are also provided.

JEL codes: C72, D43, L13, L14.

Key words and phrases: Network effects, demand-side externalities, monotone comparative statics, Cournot oligopoly, supermodularity.

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1 Introduction

It has often been observed that the nature of competition is qualitatively different in network industries. The presence of interdependencies in consumers' purchasing decisions induces demand-side economies of scale that highly affect market behavior and performance. When such effects prevail, be they of the snob or bandwagon type, purchase decisions are strongly influenced by buyers' expectations, leading to behavior not encompassed by traditional demand theory (Veblen, 1899, and Leibenstein, 1950). From an industrial organization perspective, these distinctive features raise new questions and impose some methodological challenges. In their pioneering work on markets with network effects, Katz and Shapiro (1985) proposed the concept of fulfilled expectations Cournot equilibrium, which was widely adopted in the early literature. This has led to a number of results that distinguish network markets from ordinary ones.[1]

The purpose of the present paper is to provide a thorough theoretical investigation of markets with homogeneous goods and network externalities, which unifies and extends the existing studies and tackles a number of new issues of interest that were either not previously addressed or only partially studied. We consider oligopolistic competition amongst firms in a market characterized by positive (direct) network effects when the products of the firms are perfectly compatible with each other, so that the relevant network is industry-wide. While the current literature is more concerned with the case of firm-specific networks, three arguments justify our choice. First, several important industries fit the perfect compatibility framework, in particular those in the telecommunications sector, such as fax machines and phones, but also many classical industries such as fashion and entertainment.[2] Second, there are still several outstanding issues, which, although addressed in the growing literature on network externalities, have not been fully articulated from a modeling perspective, and thus remain less than fully understood from a theoretical standpoint. Third, a good understanding of the single network case can shed quite some light on the incentives for compatibility faced by firms in the case of firm-specific networks.[3]

[1] See Economides and Himmelberg (1995), Economides (1996), Shy (2001) and Kwon (2007), among others. In contrast, the earlier literature in management science relied on explicitly dynamic models that eschewed the use of expectations (e.g. Oren and Smith, 1981 and Dhebar and Oren, 1985). More recent work using a dynamic paradigm includes Bensaid and Lesne (1996), Gabszewicz and Garcia (2007) and Doraszelski et. al. (2009), among others.

[2] In some industries, each customer may have in mind his own social network only, not the overall network, when making a purchase decision, but we follow the literature in industrial organization in ignoring this distinction.

[3] We stress that the only claim here is that the single-network case is worthy of further study. The multi-network

In its unifying scope, with an emphasis on minimal and economically meaningful assumptions on the market primitives, the paper provides a general existence result for non-trivial equilibria (i.e. those with positive production), and an extensive inquiry into the effects of market structure (or exogenous entry) on market performance. In terms of novel questions, the paper offers a general treatment of the critical issue of industry start-up, including key results on the role of the number of firms in the market and of technological improvements. It also provides some insight into the notion that the presence of expectations can substantially broaden the scope of possible outcomes relative to standard Cournot oligopoly. Throughout, the paper takes a comparative perspective in that new findings are contrasted with their standard Cournot counterparts, in an attempt to shed light on the distinctive features of (single)-network industries.

The underlying approach is to impart minimal monotonicity structure to the oligopoly model at hand, which achieves the twin goals of ensuring the existence of a fulfilled expectations Cournot equilibrium while at the same time allowing clear-cut predictions on the comparative statics of market performance with respect to the number of firms. The critical structure is imposed on the model in the form of two economically meaningful complementarity conditions on the primitives that guarantee the key properties that, along a given firm's best response, industry output increases in rivals' total output as well as in the expected network size. The overall analysis relies on lattice-theoretic methods.[4] A key benefit of the approach is to allow for more transparent economic intuition behind the cause-effect relationships we analyze.

We next provide a more detailed overview of our findings, coupled with a literature review. The problem of existence of fulfilled expectations Cournot equilibrium proceeds in two distinct steps. To establish abstract existence via Tarski's fixed point theorem, we adopt the arguments of Amir and Lambson (2000) and Kwon (2007) who directly exploit the monotonicity structure discussed above. However, as expectations about the size of the network is a key determinant of consumers' willingness to pay in these industries, the trivial, no production, equilibrium is often part of the equilibrium set. When this is the case, our previous proof of existence is not of much interest, as the underlying equilibrium may a priori be the trivial one, the presence of which can be characterized in a more direct fashion. As a consequence we complete the analysis by offering a second set of (stronger) conditions that ensure the existence of (at least) one non-trivial equilibrium, i.e. one

or imperfect compatibility cases clearly also warrant further investigation, including along the lines proposed here.

[4]See Topkis (1978), Vives (1990), Milgrom and Roberts (1990) and Milgrom and Shannon (1994).

with strictly positive sales.

Although the model is static in nature, we construct an explicit "learning dynamics", mapping consumers' expectation of the network size to the corresponding Cournot industry output to analyze the viability of the industry. This tatonnement-type dynamics is quite natural and has tacitly been the basis of many discussions of the viability issue in the literature. Studies of telecommunications markets, such as Rohlfs (1974) and Economides and Himmelberg (1995), often suggest that network industries typically have three equilibria. Under this natural dynamics, the two extreme equilibria are stable in expectations and the middle equilibrium (usually called critical mass) is unstable. The argument behind this structure is quite simple for pure network goods: If consumers expect that few buyers will acquire the good, then the good will be of little value to consumers and few of them will end up buying it. These low sales in turn further depress consumers' expectations through the above dynamics, and the market unravels towards the trivial (or no-trade) equilibrium. However, if expectations are higher to start with, other, non-trivial, equilibria will also be possible. This argument is often used to explain the start-up problem in network industries, or the difficulties faced by incumbent firms in attempting to generate enough expectations to achieve critical mass.

An important aim of the present paper is to shed light on the role of market structure as a determinant of the viability of a network industry, a novel issue that, somewhat surprisingly, has not yet been addressed in the literature. We find that the presence of more firms in the market always enhances industry viability. The same conclusion holds for exogenous technological improvements, a plausible explanation e.g. of the history of the fax machine industry.

Regarding market performance, the extremal equilibria (i.e. maximal and minimal) lead to an industry output that increases in the number of firms, n, as in standard Cournot competition. On the other hand, as this also implies an increase in the equilibrium network size or expectations, the output result does not imply that market price decreases in n. Thus, the so-called property of quasi-competitiveness, which under similar assumptions holds in standard Cournot competition, does not hold here.[5] In addition, when n increases per-firm equilibrium output increases if the demand is not too log-concave in output and decreases otherwise.

The most drastic departure from standard oligopoly lies in the effects of entry on per-firm profits. Whenever per-firm outputs and the market price increase (decrease) with n, per-firm profits increase

[5] A Cournot market is said to be quasi-competitive if the equilibrium price decreases with the number of firms.

(decrease) in n as well.[6] The conclusion that competition may increase each firm's profit is quite provocative and leads to several important implications, both from theoretical and policy-oriented perspectives. The effects of entry on industry performance as reflected in social welfare, consumer surplus and industry profits also display some distinctive features compared to standard Cournot competition. Demand-side economies of scale broaden the conditions under which social welfare increases with more entry. In addition, if the cross-effect on the inverse demand function is positive, it is possible that consumer surplus decreases with n. Katz and Shapiro (1985) explain the intuition behind this result: If the network externality is strong for the marginal consumer, then the increase in the expected network caused by the change in the number of firms will raise his/her willingness to pay for the good by more than that of the average consumer. As a consequence, the firms will be able to raise price by more than the increase in the average consumer's willingness to pay for the product and consumer surplus will fall.

The results of this paper reinforce the perception already prevalent in the literature that standard results on the workings of competition might or might not apply in network industries. As a consequence, a number of policy issues may need a fresh look and some revisiting. There may be more scope for pro-competitive cooperation or coordination by firms in network markets. One might observe a higher propensity for licensing, probably coupled with lower royalty rates or licensing fees; less patenting or a relatively more permissive attitude towards patent infringement by a firm's rivals; as well as more product standardization in industries where each firm might a priori opt for its own separate network of consumers, or for only partial compatibility of its product with rivals' products. These likely policy consequences are similar to those one might expect to see as a result of the fact that having more firms alleviates the start-up problem for the industry. In short, when more competition can be necessary to get a potential or young industry started up, or to enhance each firm's profit in an ongoing industry, the usual trade-offs between consumer surplus and producer surplus are no longer the norm, and it is not surprising that many pillars of conventional wisdom about suitable public policy for such industries might need re-examining. Proper reaction to these new incentives for coordinated action by market competitors might well require a significant overhaul of existing antitrust policy (Shapiro, 1996). In particular, it is highly desirable to arrive at a clear understanding of the respective specific market characteristics under which a given conventional

[6] This result already appears in the context of a model with an inverse demand function that is linear in output and no costs of production in Economides (1996), who in turn formalizes a remark made by Katz and Shapiro (1985).

outcome or its opposite hold. This in turn ought to rely on extensive theoretical analysis focusing on the special nature of network industries, and this is one of the motivations of the present work.[7]

Another noteworthy aspect of this paper is that we provide several explicit examples with easy closed-form solutions to illustrate in a simple way some of the underlying conclusions. In particular, Example 1 captures with precise closed-form solutions most of the relevant features often associated with the telecommunications industry in the literature.[8]

The paper is organized as follows. Section 2 presents the model, the equilibrium concept and the assumptions. Section 3 deals with existence of (non-trivial) equilibria. Section 4 studies industry viability and contains many of our fully novel results. Section 5 analyzes output, price and per-firm profits as a function of the number of firms in the market. The last section also looks at market performance as reflected in social welfare, consumer surplus and aggregate profits, again, as a function of n. Section 6 contains all the proofs of this paper. Finally, an elementary and self-contained review of the lattice-theoretic notions and results needed here forms the Appendix.

2 The analytical framework

This section presents the standard oligopoly model with network effects along with the commonly used equilibrium concept due to Katz and Shapiro (1985). In view of the more general nature of our treatment, we first enumerate all the needed assumptions we shall use later and their justification.

We consider a static model to analyze oligopolistic competition in industries with positive network effects, reflected in consumers' willingness to pay being increasing in the number of other agents acquiring the same good. The firms' products are homogeneous and perfectly compatible with each other, so there is a single network comprising the outputs of all firms in the industry.

The market is fully described by the inverse demand function $P(Z, S)$ and the number of identical firms n, each having cost function $C(x)$, where x denotes the firm's output, Z is the aggregate output in the market and S represents the expected size of the network. The cost of producing no output is zero. Considering that each consumer buys at most one unit of the good,

[7] Boone (2008) provides interesting insights into the difficulties of deriving meaningful measures of competition in regular industries. Some insights carry over to network industries.

[8] Strictly speaking, some of the examples we construct below do not satisfy all the assumptions in this paper. Since the violations are not critical in any way and analytical examples (with nice closed-form solutions that capture the features we want to highlight) are hard to come by, we are not concerned by this issue.

S also stands for the expected number of people buying the good. Sometimes, it will be useful to express the production side in terms of average cost $A(x)$, defined as $C(x)/x$ with $A(0) = C'(0)$.

For a given S, each firm's reaction correspondence is obtained by maximizing the profit function $\pi(x, y, S) = xP(x + y, S) - C(x)$

$$\widetilde{x}(y, S) = \arg\max\{\pi(x, y, S) : 0 \leq x \leq K\} \tag{1}$$

where x is the firm's level of output, y the output of the other $(n-1)$ firms in the market and $K > 0$ the production capacity of each firm.

At equilibrium, all relevant quantities x, y, Z and π will be indexed by the underlying number of firms n, e.g., we shall denote Z_n the equilibrium industry output corresponding to n firms in the market, and x_{in} the equilibrium output of firm i. When clear from the context, we will avoid the subindex i in the latter variable.

Each firm chooses its output level to maximize its profits under the assumptions that (i) consumers' expectations about the size of the network, S, is given; and (ii) the output level of the other firms, y, is fixed. Alternatively, we may think of the firm as choosing total output $Z = x + y$, given the other firm's cumulative output, y, and the expected size of the network, S, in which case, with $\widetilde{\pi}(Z, y, S) = (Z - y)P(Z, S) - C(Z - y)$

$$\widetilde{Z}(y, S) = \arg\max\{\widetilde{\pi}(Z, y, S) : y \leq Z \leq y + K\}. \tag{2}$$

Consistency requires $\widetilde{Z}(y, S) = \widetilde{x}(y, S) + y$.

An equilibrium in this game is a vector $(x_{1n}, x_{2n}, ..., x_{nn})$ that satisfies the following conditions

1. $x_{in} \in \arg\max\left\{xP\left(x + \sum_{j \neq i} x_{jn}, S\right) - C(x) : 0 \leq x \leq K\right\}$; and
2. $S = \sum_i x_{in}$.

Since the seminal paper by Katz and Shapiro (1985), this notion of equilibrium, known as "Fulfilled Expectations Cournot Equilibrium (FECE)," has been used for oligopolies with network effects. It requires that both consumers and firms correctly predict the market outcome, so that their beliefs are confirmed in equilibrium. While strategic in their choice of outputs in the usual Cournot sense, firms are "network-size taking" in their perceived inability to directly influence customers' expectations of market size. One plausible justification for this is that firms are unable to credibly commit to output levels that customers could observe and reliably use in formulating expectations about network size (Katz and Shapiro, 1985).[9] Naturally, the plausibility of the FECE concept increases with the number of firms present in the market.[10]

Viewing S as an inverse demand shift variable, condition 1 just describes the equilibrium in standard Cournot competition with exogenous S. Let $z_n(S)$ denote the corresponding industry output equilibrium correspondence. Adding condition 2, an aggregate output $Z_n \in z_n(S)$ constitutes a FECE industry output if it satisfies $Z_n = S$ as well. As a consequence, if we graph $z_n(S)$ as a function of S, the FECE industry outputs are all the points where this correspondence crosses the 45° line. This idea will play a key role in both the existence proof and the viability analysis.

Another, fully game-theoretic, interpretation of this equilibrium notion is in the context of a two-stage game, wherein a market maker (or a regulator) announces an expected network size S in the first stage, and firms compete in Cournot fashion facing inverse demand $P(Z, S)$ in the second stage. If the market maker's objective function is to minimize $|S - z_n(S)|$, then to any subgame-perfect equilibrium of this game corresponds a FECE of the Cournot market with network externalities, and vice-versa.

Whenever well-defined, we denote the maximal and minimal points of a set by an upper and a lower bar, respectively. Thus, for instance, $\overline{Z}_n$ and $\underline{Z}_n$ are the highest and lowest industry equilibrium outputs when there are n firms in the market.

Denote by $W(Z, S) \triangleq \int_0^Z P(t, S)\,dt - ZA(Z/n)$ the Marshallian social welfare when aggregate output is Z, all firms produce the same quantity and the expected size of the network is S. Similarly, consumer surplus is $CS(Z, S) \triangleq \int_0^Z P(t, S)\,dt - ZP(Z, S)$.

We now list the assumptions used in this paper, starting with a set of standard ones, followed by more substantive conditions.

The standard assumptions are

(A1) $P(.,.)$ is twice continuously differentiable, $P_1(Z, S) < 0$ and $P_2(Z, S) > 0$.

(A2) $C(.)$ is twice continuously differentiable and increasing.

[9] Were such commitment credible for firms, standard Cournot equilibrium with inverse demand $P(Z, Z)$ would be a more appropriate concept. A direct comparison between these two concepts appears in Katz and Shapiro (1985).

[10] A well-known parallel is the fact that the price-taking assumption of perfect competition is more plausible in markets with many producers, and thus more diffuse competition.

(A3) $x_i \leq K$, for all firm i.

These are all commonly used assumptions, including $P_2(Z,S) > 0$, which reflects positive network effects, or the property that consumers' willingness to pay increases in the expected number of people who will buy the good. Assumption A3 imposes capacity constraints on the production process of each firm, a convenient condition to force compact output sets in a setting where firms may otherwise wish to produce unbounded output levels. Our results do not rely in any way on K taking on any particular set of values, as in Amir and Lambson (2000).

The second set of assumptions are placed on two functions that play a key role in the overall analysis. Let $\Delta_1(Z,y)$ denote the cross-partial derivative of $\widetilde{\pi}(Z,y,S)$ with respect to Z and y, and $\Delta_2(Z,S)$ the cross-partial derivative of $\log P(Z,S)$ with respect to Z and S, scaled by $[P(Z,S)]^2$,

$$\begin{aligned}\Delta_1(Z,y) &= -P_1(Z,S) + C''(Z-y) \text{ and}\\ \Delta_2(Z,S) &= P(Z,S)\,P_{12}(Z,S) - P_1(Z,S)\,P_2(Z,S).\end{aligned}$$

The domains of Δ_1 and Δ_2 are $\varphi_1 \equiv \{(Z,y) : y \geq 0, Z \geq y\}$ and $\varphi_2 \equiv \{(Z,S) : Z \geq y, S \geq 0\}$ respectively, both of which are lattices (in the product order).

The second set of assumptions is

(A4) $\Delta_1(Z,y) = -P_1(Z,S) + C''(Z-y) > 0$ on φ_1.

(A5) $\Delta_2(Z,S) = P(Z,S)\,P_{12}(Z,S) - P_1(Z,S)\,P_2(Z,S) > 0$ on φ_2.

Assumption A4 allows for limited scale economies in production, and has been justified in detail by Amir and Lambson (2000). In terms of the model structure, Assumptions A4 and A5 guarantee that the profit function $\widetilde{\pi}(Z,y,S)$ has strictly increasing differences on φ_1 and the strict single-crossing property in $(Z;S)$, respectively, so that $\widetilde{Z}(y,S)$ increases in y and S, respectively.

The key novel assumption here is A5, which has the precise economic interpretation that the elasticity of demand increases in the expected network size S.[11] In his pioneering study of the elementary microeconomic foundations of interdependent demands, Leibenstein (1950) suggested that demand is more elastic in network markets because individual reactions to price changes are followed by additional reactions, in the same direction, to each other's change in consumption. A5

[11] The price elasticity of demand is $-\left(\frac{\partial P(Z,S)}{\partial Z}\frac{Z}{P(Z,S)}\right)^{-1} = -\left(Z\frac{\partial \log P(Z,S)}{\partial Z}\right)^{-1}$, which is increasing in S if and only if $\log P(Z,S)$ has increasing differences in (Z,S) (Topkis, 1998, p. 66).

essentially captures the cumulative effect of these mutually reinforcing effects on aggregate demand. Another plausible interpretation of A5 is that it provides a formalization of the concept of demand-side scale economies that is often postulated as a characteristic of network effects, though not in a precise manner. A5 also embodies a key respect in which the present paper departs from the static literature, much of which deals with the case of additive network effects (Katz and Shapiro, 1985, Economides and Himmelberg, 1995, and Economides, 1996, among others).[12]

3 Existence of equilibrium

In this section we provide a general equilibrium existence result, exploiting the minimal monotonic structure of the model reflected in A4-A5. As the trivial (zero-output) equilibrium is often part of the equilibrium set, we derive a second result that offers additional conditions to guarantee the existence of a non-trivial equilibrium, i.e. one with strictly positive industry output.

We begin with the central monotonicity result, which is a direct consequence of A4 and A5.

Lemma 1 *Assume A1-A5 are satisfied. Then, every selection of the best-response correspondence* $\widetilde{Z}(y, S)$ *is increasing in both* y *and* S.

This lemma leads to an abstract existence result for symmetric equilibrium, along with the fact that the same assumptions preclude the possibility of asymmetric equilibria.

Theorem 2 *Assume A1-A5 are satisfied. Then, for each* $n \in N$*, the Cournot oligopoly with network effects has (at least) one symmetric equilibrium and no asymmetric equilibria.*

The monotonicity structure behind this existence result will turn out to also drive most of the key results of this paper, most of which have a comparative statics flavor. Comparing A1-A5 with the set of assumptions in standard Cournot competition, the only new requirement is that the price elasticity of demand increases with the network size, A5, taking $P_2(Z, S) > 0$ as a natural property of network markets. Analogs of all other assumptions are also needed for proving existence in the standard Cournot model, as reflected in [Amir and Lambson (2000), Theorem 2.1] reproduced next.

[12] An exception to this classification is Resende and Laussel (2008), which deals in a novel way with multiplicative network effects in a dynamic setting.

Lemma 3 *Assume A1-A4 are satisfied. Then, for each $n \in N$, (i) the standard Cournot oligopoly (with exogenous S) has a symmetric equilibrium and no asymmetric equilibria; (ii) if A5 also holds, the extremal selections of $z_n(S)$, $\overline{z}_n(S)$ and $\underline{z}_n(S)$, increase in S; and (iii) if in addition A5 holds and $P(Z,S)$ is log-concave in Z, $z_n(.)$ is a single-valued and continuous function.*

In Section 2 we noted that a fixed point of $z_n(S)$ constitutes a FECE industry output. As a consequence, statements (i) and (ii) in the last lemma could be used to show equilibrium existence through Tarski's Theorem. Although less direct than the approach behind Theorem 2, this idea plays a key role in the proof of existence of a non-trivial equilibrium.

It is well-known that in network markets the trivial (zero-production) outcome is often an equilibrium. This phenomenon intensifies when the network good has little stand-alone value, i.e. $P(Z,0)$ is small. Given any such good, if end users believe no one else will acquire it, the good will have no value, and the trivial outcome will necessarily be part of the equilibrium set. Telecommunications industries, such as fax, phone and e-mail, typically exhibit this characteristic.

In such markets, Theorem 2 is not of much interest since the underlying equilibrium may a priori be the trivial one. To complete the analysis, we give a simple characterization of the trivial equilibrium and then add extra assumptions to ensure the existence of a non-trivial equilibrium.

Lemma 4 *The trivial outcome is an equilibrium if and only if $xP(x,0) \leq C(x)$ for all $x \in [0,K]$.*

This lemma simply says that the trivial outcome is part of the equilibrium set if and only if when the common expectation (amongst firms and consumers) about the size of the network is zero, and a firm believes the other firms will produce no output, the best it can do under the required condition is to produce zero as well. The proof follows directly from the definition of FECE.

To provide additional sufficient conditions to ensure the existence of a non-trivial equilibrium (with strictly positive industry output), we need to make use of a fictitious objective function that is known to achieve its maximum at a Cournot equilibrium industry output level, for given S. Define

$$\Pi(Z,S) \triangleq \frac{n-1}{n}\left[\int_0^Z P(t,S)\,dt - nC(Z/n)\right] + \frac{1}{n}\left[ZP(Z,S) - nC(Z/n)\right]. \qquad (3)$$

For given S, the function (3) $\Pi(Z,S)$ is a weighted combination of welfare and industry profits, with respective weights $\frac{1}{n}$ and $\frac{n-1}{n}$, as given in Bergstrom and Varian (1985) for standard Cournot.

Theorem 5 *Assume A1-A5 are satisfied. Then, there exists a non-trivial equilibrium if at least one of the following conditions is also satisfied* *(i) zero is not an equilibrium output (i.e. the condition of Lemma 4 does not hold);*

(ii) zero is an equilibrium output, $P(0,0) = C'(0)$*,* $n > [-P_1(0,0) + C''(0)] / [P_1(0,0) + P_2(0,0)]$ *and* $P_1(0,0) + P_2(0,0) > 0$*; or*

(iii) zero is an equilibrium output, $C''(.) \geq 0$ *and* $\Pi(Z', S) \geq \Pi(Z, S)$ *for some* $S \in (0, nK]$*, some* $Z' \geq S$ *and all* $Z \leq S$*.*

Theorem 2 ensures equilibrium existence. Hence, if 0 is not part of the equilibrium set, there must be an equilibrium with a strictly positive industry output, and Theorem 5(i) follows. This applies only to network goods with sufficiently high stand-alone value (cf. Lemma 4).

The extra requirements in (ii) guarantee that, although $z_n(0) = 0$, $z_n(S)$ starts above the 45° line near 0. The existence of a non-trivial equilibrium follows by Lemma 3 (ii). Formally, this derives from applying Tarski's Theorem to $z_n(S)$ for $S \in [\epsilon, nK]$, for some $\epsilon > 0$ small enough. As expected, the stronger the network effect around the origin is, as captured by $P_2(0,0)$, the less stringent the existence condition for the non-trivial equilibrium gets, i.e. the lower the threshold value of n is.

Condition (iii) ensures that, although $z_n(0) = 0$, $z_n(S)$ is above the 45° line at some $S \in (0, nK]$, so a non-trivial equilibrium exists by Tarski's Theorem applied to $z_n(.)$ mapping $[S, nK]$ to itself. The interpretation of the inequality $\Pi(Z', S) \geq \Pi(Z, S)$ derives from the meaning of $\Pi(Z', S)$ as a welfare-profit combination (see also Bergstrom and Varian, 1985).

The proof of Theorem 5 uses the following intermediate result, which also plays a key role in the viability analysis (Section 4).

Lemma 6 *Assume A1-A5 are satisfied. If* $0 \in z_n(0)$*, then* $z_n(0) = 0$*, i.e.* $z_n(0)$ *is single-valued. If in addition* $P(0,0) = C'(0)$*, the slope of* $z_n(.)$ *is also single-valued and right-continuous at* 0*, and*

$$z_n'(0) = \frac{nP_2(0,0)}{-(n+1)P_1(0,0) + C''(0)}. \tag{4}$$

If the trivial equilibrium is not interior, i.e. $P(0,0) < C'(0)$*, then* $z_n'(0) = 0$*.*

Thus, though $z_n(.)$ is a correspondence, when 0 is part of the equilibrium set, it is single-valued at the origin. If in addition the trivial equilibrium is interior, the slope of this function is given by (4) and depends on n. This observation plays a key role in the analysis of industry viability.

The possibility of multiple equilibria in markets with network effects is more of a norm than an exception. Multiple equilibria are due to the positive feedback that derives from expectations: If consumers believe the good will not succeed, it will usually fail. On the contrary, if they expect it to succeed, it usually will.

Amir and Lambson (2000) show that log-concavity of $P(Z,S)$ in Z is sufficient for uniqueness in standard Cournot competition (along with A4). The added benefit of this condition here would thus be to ensure that $z_n(.)$ is single-valued and continuous, as shown in Lemma 3. When network effects prevail, much stronger conditions are required to ensure uniqueness. Since our methodology allows us to deal with multiple equilibria, we leave this issue aside.

4 Industry viability

This section provides an extensive treatment of industry viability, via (i) a formalization of expectations dynamics and the associated stability analysis of FECE, and (ii) two key results on the effects of exogenously changing market structure and technological improvements on industry viability. As such, it contains much analysis with little formal counterpart in the extant literature.

Many studies suggest that the left panel of Figure 1 reflects the structure of specific telecommunications industries. The underlying game there displays three possible equilibria, the trivial equilibrium, a middle unstable equilibrium, usually called critical mass, and a high stable equilibrium.[13] The intuition behind this configuration is quite simple: If all the consumers expect that no one will acquire the good, then the good has no value and no one will end up buying it, resulting in the trivial equilibrium for the industry. However, if expectations are higher to start with, another, non-trivial, equilibria will prevail.

Whenever the trivial equilibrium is locally stable in expectations (as in Figure 1), the market will never emerge as a result of an expected network size that is too low to start with. In view of the equilibrium concept adopted here, the incumbent firms are simply unable to influence these expectations to get them past the critical mass. Under such conditions, even if the industry does get going, Cournot equilibrium on the basis of small expectations cannot lead firms to produce enough output to generate prospects beyond the critical mass, and the industry will unravel through a natural process towards the trivial equilibrium. This argument is commonly invoked to capture the

[13] There are several definitions of the notion of critical mass in the literature, some in dynamic settings and others in static settings. In the present paper, we wish to adapt the most common definition, which is as the smallest non-zero (Cournot-) unstable FECE, to our framework taking into account the multi-valuedness of $z_n(S)$.

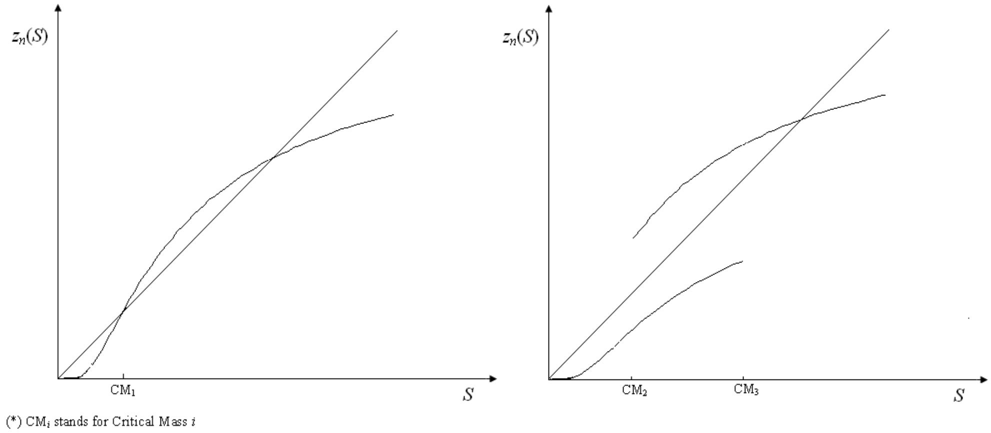

Figure 1: Viability and Basin of Attraction of the Trivial Equilibrium

start-up problem that frequently affects these markets, and is often referred to as the "chicken and egg" paradox. Oren and Smith (1981) offer an early discussion of this phenomenon in electronic communications markets.

The tacit dynamic process underlying this analysis can be formalized through the following expectations/network size recursion, starting from any initial $S_0 \geq 0$,

$$S_k = \widehat{z}_n \left(S_{k-1}\right), \; k \geq 1 \tag{5}$$

where $\widehat{z}_n$ will denote either the maximal or minimal selection of $z_n(.)$ throughout (except where otherwise indicated).

This process thus begins with a historically given initial expectation S_0, then postulates that firms react by engaging in Cournot competition with demand $P(Z, S_0)$, leading to an industry output $\widehat{z}_n\left(S_0\right)$. The latter will in turn determine consumers expectation $S_1 = \widehat{z}_n\left(S_0\right)$, and the process repeats indefinitely. This yields a sequential adjustment course in which consumers and firms behave myopically with respect to the size of the network. Taking a single-valued selection of $z_n(.)$ amounts to selecting one particular Cournot equilibrium for each given S.

For each extremal selection of $z_n(.)$, we can formally define the corresponding critical mass as the smallest initial expectation $\widehat{S}_0$ such that for all $S_0 > \widehat{S}_0$, the orbit given by (5) converges to a nonzero FECE. This definition captures the notion of critical mass irrespective of whether the selection at hand is continuous, or continuous from one side only (i.e. right or left), or neither, at that specific point.[14] We next formalize our notion of stability.

Definition 7 *The trivial equilibrium is stable if there is a right neighborhood V of 0 such that for all S_0 in V, the orbit $S_k = \widehat{z}_n(S_{k-1}) \to 0$ as $k \to \infty$.*

It is intuitive that the maximal (minimal) selection is most (least) favorable for the viability of the industry. Let V_n denote the largest set of values of S_0 for which the trivial equilibrium is stable, i.e. V_n is the basin of attraction of the trivial equilibrium. In view of Lemma 6, when zero is a FECE aggregate output, $\widehat{z}_n(.)$ is continuously differentiable at 0. Assuming henceforth that this derivative is (generically) not equal to 1, 0 is an isolated fixed-point (for a formal proof, see e.g., Granas and Dugundji, 2003, pp. 326-327). Since in addition $\widehat{z}_n(.)$ is increasing in S, V_n is an interval.[15] In the left panel of Figure 1, where $z_n(.)$ is single-valued, there is only one interval equal to $(0, CM_1)$; in the right panel of that figure the highest and lowest selections of $\widehat{z}_n$ induce $V_n = (0, CM_2)$ and $V_n = (0, CM_3]$, respectively.[16]

Each industry can be classified into one of three possible categories in terms of viability.[17]

Definition 8 *An industry is said to be (i) uniformly viable if every orbit in (5) converges to some non-zero equilibrium starting from any $S_0 > 0$; (ii) conditionally viable if the convergence in (i) takes place only from sufficiently high S_0; and (iii) nonviable if every orbit in (5) converges to 0 from any $S_0 \geq 0$.*

The next observation provides sufficient conditions for each possible viability outcome by linking it to our previous result on the existence of a non-trivial equilibrium (being similar to that of Theorem 5, the proof is omitted).

Corollary 9 *Assume A1-A5 are satisfied. An industry is (i) uniformly viable if and only if either condition (i) or (ii) of Theorem 5 holds; (ii) conditionally viable if condition (iii) of Theorem 5 holds; and (iii) nonviable if the conditions of Lemma 4 holds and this equilibrium is unique.*

[14] Related issues are addressed in some detail in Echenique (2002).

[15] Since z_n is u.h.c., $\underline{z}_n = \min z_n$ is l.s.c. and left-continuous, and $\overline{z}_n = \max z_n$ is u.s.c. and right-continuous. Hence, V_n is open at its upper bound for $\overline{z}_n$ while it may or may not be when we consider $\underline{z}_n$.

[16] The fact that $(0, CM_2)$ is open and $(0, CM_3]$ is right-closed follows from both Figure 1 and Footnote 15.

[17] This definition extends in the obvious way to any increasing selection of $z_n(S)$, i.e. not only the extremal ones. Thus, for any increasing selection $\widehat{z}_n(S)$, the critical mass is 0 if the industry is uniformly viable, ∞ if it is nonviable, and satisfies $\widehat{S}_0 > 0$ and $\lim_{S \uparrow \widehat{S}_0} \widehat{z}_n(S) \leq \widehat{S}_0 \leq \lim_{S \downarrow \widehat{S}_0} \widehat{z}_n(S)$ if the industry is conditionally viable.

To provide a basis for comparing two different situations that might prevail for the same industry, we need to formalize a partial order for increasing viability.

Definition 10 *The viability of an industry increases if either (i) the industry goes from nonviable to conditionally viable, or from the latter to uniformly viable; or (ii) the industry is conditionally viable and V_n contracts.*

Clearly, viability depends on the equilibrium selection under which the industry operates. It is worth noting that, in contrast to standard Cournot industries, firms here would not necessarily prefer the lowest equilibrium selection out of $z_n(.)$ since that selection may lead to a lower viability for the industry (and thus to zero profits from more initial values of S_0) than the largest selection. Likewise, consumers would always prefer the largest selection of $z_n(.)$ in standard Cournot markets, but not necesarily here.

The next result, a key finding of this paper, shows that additional firms in the market and/or a technological improvement always enhance the viability of a network industry.[18] We capture exogenous technological change by a decrease in α for the cost function $\alpha C(.)$.[19] Examples 1 and 2 below illustrate these important effects.

Theorem 11 *Assume A1-A5 are satisfied. Then,*

(i) more firms in the market and/or technological improvements always increase the viability of the industry (i.e. $\widehat{z}_n(.)$ shift up as n increases and/or α decreases); and

(ii) if the trivial outcome is an equilibrium (i.e. the condition of Lemma 4 holds) and $P_1(0,0) + P_2(0,0) \leq 0$, an industry cannot be uniformly viable for any n.[20]

Theorem 11 captures the key role of market structure in industry viability: having more firms around implies a lower critical mass would be needed to launch a given industry. The underlying intuition is intimately connected to the FECE concept. Consider the natural question: In case S_0 happens to be below the critical mass, what prevents the existing firms from attempting to act as

[18] Economides and Himmelberg (1995) show that, under some conditions, market structure has no effect on the critical mass. Our results do not coincide because we define critical mass in a different way.

[19] The result would extend to more general (non-uniform) shifts in the cost function, provided both the industry cost and the marginal cost functions shift downwards. The proof is essentially the same.

[20] This statement holds even if we consider the highest selection of $z_n(.)$.

if there were more of them by producing a higher output level in an effort to influence consumers' expectations of the network size upwards? In a context where the appropriate solution concept is FECE, firms presumably cannot commit to their desired output levels in a credible way, and, likewise, attempting to inflate their number by committing to a higher output would also not be credible, and would thus not constitute behavior compatible with the FECE concept.

In industries with multiple firms having their own versions of the same general good, this result might explain why firms often settle for full compatibility between their products, instead of incompatibility. Their objective is to generate a single industry network that would be viable, when separate networks with one firm each would not be. This implies that some form of cooperation amongst direct rivals could be needed for their products to succeed. One example is the case of Sony and Philips, who jointly created industry standards for compact disc in the mid 80's (Shapiro, 1996). Such forms of cooperation have no counterparts in non-network markets.

The last theorem also captures the fundamental effect of an exogenous technological change on industry viability. A technological improvement also lowers the critical mass that would be needed to start the market up. This key and intuitive result can shed some light on observed market behavior. The fax market took decades beyond the discovery of the initial technology to get started (Shapiro and Varian, 1998). Now and then, an attempt at launching a new product with network effects is seen to fail. One plausible diagnosis according to the present analysis is that the product might be too costly at the early stages of the emerging industry.[21]

Example 1. Consider the symmetric Cournot oligopoly with no production costs, and inverse demand function given by

$$P(Z,S) = \exp\left(-\frac{2Z}{\exp(1-1/S)}\right) \text{ with } Z,S \in [0,nK].$$

The reaction function of a firm is $\widetilde{x}(y,S) = (1/2)\exp(1-1/S)$. Since each firm has a dominant strategy, $\widetilde{x}(y,S)$ does not depend on y, and we can add the reaction functions to obtain $z_n(S) = (n/2)\exp(1-1/S)$.

An equilibrium industry output solves $z_n(Z) = Z$ in Z. Then we have: $Z_1 = \{0\}$, $Z_2 = \{0,1\}$, $Z_3 = \{0, 0.457, 2.882\}$ and $Z_4 = \{0, 0.373, 4.311\}$, as shown in Figure 2.

[21] The quality of the production technology can also be a key factor in determining start-up success, but in the present context this can only be captured partly via the cost function.

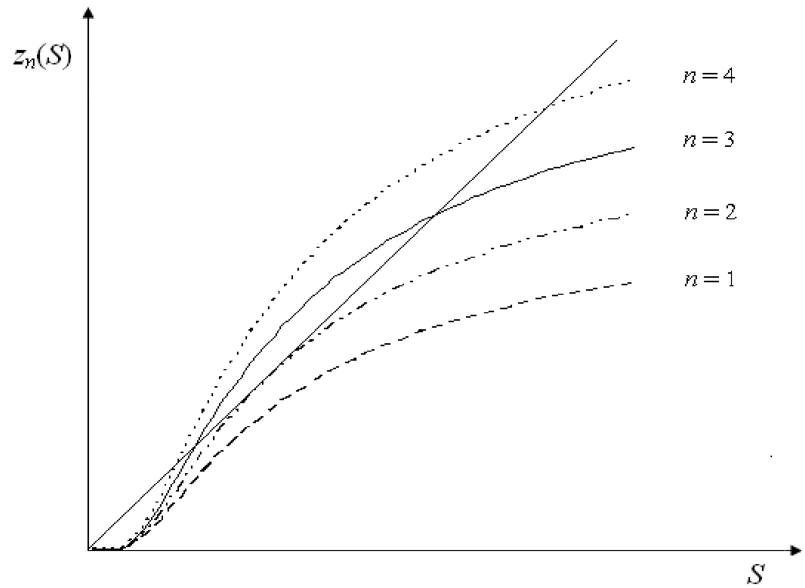

Figure 2: Viability and Market Structure

As can be easily seen, the trivial equilibrium is always stable. With only one firm in the market, this is the only equilibrium, so the industry is nonviable. With one extra firm, a larger equilibrium emerges and the industry becomes conditionally viable (barely, since $z_n(.)$ is tangent to the 45° line). For a larger number of firms, the equilibrium configuration encompasses three equilibria; the two extreme ones are stable and the intermediate one is unstable. The unstable equilibrium, often called critical mass, decreases in n. This is an exact closed-form example of the three-equilibrium constellation that is often portrayed as typical in many network industries.

Here, $z_n(.)$ shifts up as n increases. The industry goes from nonviable to conditionally viable as n goes from 1 to 2 firms. As n further increases, viability increases since the basin of attraction of 0 shrinks, but uniform viability is never attained as $P_1(0,0) + P_2(0,0) = 0$ (cf. Theorem 11). □

In our first example, initial expectations must be high enough to start the market up (when $n \geq 2$). Although the critical mass shrinks as the number of firms increases, the start-up problem always persists. The next example shows an extreme case where this problem disappears with a technological improvement.

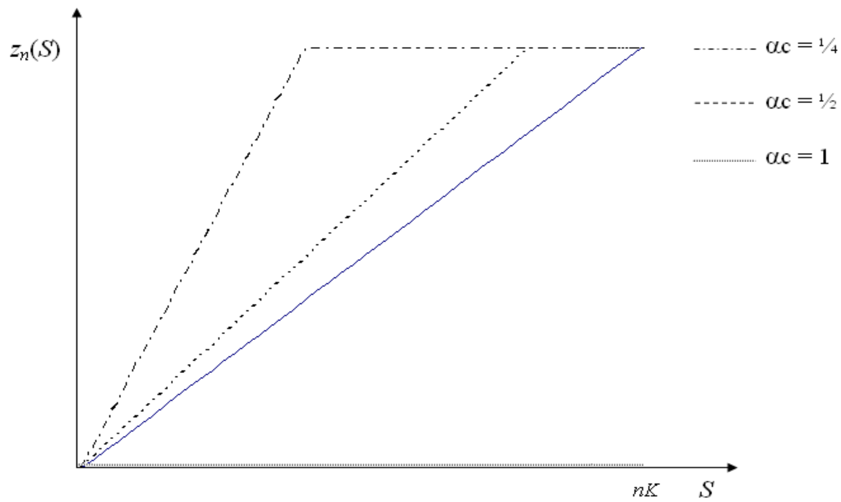

Figure 3: Viability and Technological Improvements

Example 2. Consider an oligopoly with cost function $C(x) = \alpha cx$, $c > 0$, and

$$P(Z,S) = \exp\left(-\frac{Z}{bS}\right) \text{ with } Z, S \in [0, nK] \text{ and } b > 0.$$

Here α captures technological improvements. The profit function is

$$\pi(x,y,S) = \left[\exp\left(-\frac{x+y}{bS}\right)\right] x - \alpha cx.$$

The reaction function of any given firm is implicitly defined by

$$\left[\exp\left(-\frac{\widetilde{x}(y,S)+y}{bS}\right)\right]\left(1-\frac{\widetilde{x}(y,S)+y}{nbS}\right) - \alpha c = 0 \tag{6}$$

when $\widetilde{x}(y,S)$ is interior. When the left hand side of (6) is negative for all $\widetilde{x} \in [0,K]$ then $\widetilde{x}(y,S) = 0$, and it takes the value K when the left hand side of (6) is positive for all $\widetilde{x} \in [0,K]$.

The aggregate equilibrium output with exogenous S, $z_n(S)$, is implicitly defined by

$$\left[\exp\left(-\frac{z_n(S)}{bS}\right)\right]\left(1-\frac{z_n(S)}{nbS}\right) - \alpha c = 0 \tag{7}$$

when $z_n(S)$ is interior. Otherwise, $z_n(S)$ is either 0 or nK.

Figure 3 illustrates the FECE for $n = 10$ and $b = 2$, given three possible values of αc: 1/4, 1/2 and 1. When the technology is costly, $\alpha c = 1$, $z_n(S)$ is 0 for all S. Then the trivial equilibrium is the unique FECE, and the industry is nonviable. After technological improvements, $\alpha c = 1/2$ and

$\alpha c = 1/4$, every orbit in (5) converges to nK starting from any $S_0 > 0$, so the industry becomes uniformly viable. □

The first example illustrates an interesting situation where the presence of network effects might have unusual implications on firms' attitudes towards intellectual property rights and entry deterrence. Indeed, firms will not be tempted to engage in entry deterrence activities if their number is insufficient to start the market up. In such a case, those in possession of patents will have a much higher than usual incentive to engage in licensing to their rivals on rather generous terms. Naturally, such generosity will prevail only until the industry is started up, or until profits cease to increase with the number of competitors, as we shall see below.

The last example shows the key effect of technological improvements on the viability of the industry. Even with a large number of potential competitors, the market might not start up until the industry manages to lower production costs sufficiently.

5 Exogenous entry and market performance

This section studies the effects of market structure (or exogenous entry) on the equilibrium industry output, per-firm output, market price, per-firm profits, consumer surplus and social welfare. Amir and Lambson (2000) and Amir (2003) address similar questions for standard Cournot competition, and show that scale economies can lead to counterintuitive results about the most basic aspects of market response to increased competition. We show next that, under network effects, similar reversals are typically much easier to come by, and that they can be generated solely by demand-side externalities instead of production scale economies. In other words, they derive from increasing returns on the demand side of the market, rather than the supply side. We will provide either sufficient conditions for these reversals when appropriate, or otherwise at least closed-form examples illustrating these possibilities.

The analysis that follows makes all the statements on the largest equilibrium, i.e. the one with the largest equilibrium outputs, namely, $\overline{Z}_n$ and $\overline{x}_n$. When the zero outcome is an equilibrium, it is clearly the smallest equilibrium. Since it is invariant in the number of firms, its comparative statics questions are trivial. When the zero outcome is not part of the equilibrium set, our conclusions also apply to the minimal selections, $\underline{Z}_n$ and $\underline{x}_n$.

Define the interval of output levels $I_n \triangleq [\overline{Z}_n, \overline{Z}_{n+1}]$. The first result relates entry to equilibrium industry output and market price.

Theorem 12 *Assume conditions A1-A5 are satisfied. Then, we have*

(i) the extremal equilibrium industry outputs, $\overline{Z}_n$ and $\underline{Z}_n$, increase in n; and

(ii) if $P_1(Z,Z) + P_2(Z,Z) \geq (\leq) \, 0$ on I_n, then $P\left(\overline{Z}_{n+1}, \overline{Z}_{n+1}\right) \geq (\leq) \, P\left(\overline{Z}_n, \overline{Z}_n\right)$.

Theorem 12 (i) is also true in standard Cournot competition, as shown by [Amir and Lambson (2000), Theorem 2.2 (b)]. In the latter, the usual law of demand suffices for the market price to decrease after new entry. As Theorem 12 (ii) indicates, the effect of entry on market price is ambiguous when network effects prevail. The reason is that when industry output increases the firms must set the price low enough to attract the marginal consumer, but when more buyers join the network consumers' willingness to pay increases. Thus the overall effect of entry on the market price depends on how strong the output effect is, relative to the network effect. As a consequence, the so-called property of quasi-competitiveness, which under similar assumptions holds in the standard Cournot game, need not be satisfied here.

For the effects of entry on per-firm outputs and profits, a new function is needed (some insight into its sign is given later)

$$g(Z) = \left[P(Z,Z) - C'(Z/n)\right]\left[P_{11}(Z,Z) + P_{12}(Z,Z)\right] - P_1(Z,Z)\left[P_1(Z,Z) + P_2(Z,Z)\right]. \qquad (8)$$

Theorem 13 *In addition to A1-A5, assume $\overline{Z}_n$ and $\overline{Z}_{n+1}$ are interior equilibria. Then, we have*

(i) if $g(Z) \geq 0$ on I_n, the largest per-firm equilibrium output increases in n, i.e. $\overline{x}_{n+1} \geq \overline{x}_n$; and

(ii) if $g(Z) \leq 0$ on I_n, the largest per-firm equilibrium output decreases in n, i.e. $\overline{x}_{n+1} \leq \overline{x}_n$.

In short, this result means that the scope for the business-stealing effect, which is nearly universal in standard Cournot oligopoly, is quite a bit narrower in the presence of network externalities. On the other hand, the scope for the opposite, or business-enhancing, effect is much broader in the present setting, as we see next.

Corollary 14 *In addition to the assumptions of Theorem 13, assume no costs of production. Then $\overline{x}_{n+1} \geq \overline{x}_n$ if, for $Z \in I_n$,*

$$\left[P(Z,Z)P_{12}(Z,Z) - P_1(Z,Z)P_2(Z,Z)\right] + \left[P(Z,Z)P_{11}(Z,Z) - P_1^2(Z,Z)\right] \geq 0. \qquad (9)$$

The left-hand side of (9) is the same as $g(Z)$ when the firms face no production costs. Its first term is positive by A5, and log-convexity of $P(Z,S)$ in Z ensures the second one is positive as well. Therefore log-convexity is a sufficient, but not necessary, condition for the highest per-firm equilibrium output to increase after new entry whenever marginal costs are zero. Amir and Lambson (2000), Theorem 2.3, require log-convexity to globally ensure the same result for standard Cournot competition. Hence network effects facilitate this unusual outcome.

Based on Theorems 12 and 13, the following result deals with the effects of entry on per-firm equilibrium profits. Recall that in standard Cournot oligopoly, the only part of the conventional wisdom about the effects of competition that is universally valid is that per-firm profits decline with the number of competitors (Amir and Lambson, 2000, and Amir, 2003). We now show that in the presence of network effects, this result can be easily reversed.

Theorem 15 *In addition to A1-A5, assume $\overline{Z}_n$ and $\overline{Z}_{n+1}$ are interior equilibria. Then, we have*

(i) if $P_1(Z,Z) + P_2(Z,Z) \geq 0$ and $g(Z) \geq 0$ on I_n, at the largest equilibrium, $\pi_{n+1} \geq \pi_n$; and

(ii) if $P_1(Z,Z) + P_2(Z,Z) \leq 0$ and $g(Z) \leq 0$ on I_n, at the largest equilibrium, $\pi_{n+1} \leq \pi_n$.

The first result provides sufficient conditions for the firms in the market to prefer further entry by new firms. It generalizes a result in Economides (1996), based on a more specific formulation, which in turn formalizes a remark made by Katz and Shapiro (1985).

Although surprising, the intuition for this outcome is simple. New entry increases the equilibrium industry output, as shown in Theorem 12, and a direct effect is that market price goes down by the usual law of demand. But via the effect on the size of the network, this output increase also shifts the inverse demand function up, thus pushing for a price increase. Then, if the overall effect on the market price is positive and each firm increases own output, the existing firms in the market are better-off after new entry. As Economides (1996) states, if the externalities are strong, the network effect dominates the usual competitive effect of entry.

A natural question arises when profits increase in n. Why can't the existing firms attempt to act as if there were more of them in order to each get higher profits at equilibrium? Since they would do so by producing a higher output level in an effort to influence consumers' expectations of the network size upward, the answer is the same as for the start-up problem: the tacit lack of commitment power on the part of the firms, which is at the heart of the FECE concept.

The next result follows as a simple corollary of our last theorem. Its extra requirement captures (as a special case) one of the conditions often imposed in the network models: no second order effects on the inverse demand function.

Corollary 16 *In addition to the conditions of Theorems 13 and 15, assume* $P_{11}(Z,Z)+P_{12}(Z,Z)=0$*, for all* Z*. If* $P_1(Z,Z)+P_2(Z,Z)\geq(\leq)\,0$ *on* I_n*, then, at the largest equilibrium,*

(i) per-firm equilibrium output increases (decreases) in n*, i.e.* $\overline{x}_{n+1}\geq(\leq)\,\overline{x}_n$*; and*

(ii) per-firm equilibrium profits increase (decrease) in n*, i.e.* $\pi_{n+1}\geq(\leq)\,\pi_n$*.*

The new condition here, $P_{11}(Z,Z)+P_{12}(Z,Z)=0$, is satisfied if, for example, $P(Z,S)=h(S)-kZ$ with $h(.)$ an increasing function, or $P(Z,S)=f(S-Z)$ with $f(.)$ increasing.

The next example highlights the implications of Theorem 15.

Example 3. Consider a Cournot oligopoly with no production costs and

$$P(Z,S)=\max\{a+bS^{\alpha}-Z,0\}\ \text{ with } Z,S\in[0,nK],\ a\geq 0,\ b>0 \text{ and } \alpha\in(0,1).$$

The reaction function of any given firm is $\widetilde{x}(y,S)=\max\{(a+bS^{\alpha}-y)/2,0\}$. (Here we assume K is large enough.) After a simple computation, the symmetric equilibrium industry output is implicitly defined by $-Z_n(1+n)+na+nbZ_n^{\alpha}=0$.

Let $a=10$, $b=5$ and $\alpha=4/5$. Upon computation, per-firm equilibrium profits for different values of n are

$$\begin{aligned}\pi_1 &\approx 14,561<\pi_2\approx 49,255<\pi_3\approx 67,316<\pi_4\approx 70,676\\ \pi_5 &\approx 67,288>\pi_6\approx 61,520>\pi_7\approx 55,301>\pi_8\approx 49,404>...>\pi_{21}\approx 14,444.\end{aligned}$$

We observe that when the number of firms is small, $n=1,2$ or 3, incumbent firms are better off if an extra firm enters the market. When $n\geq 4$, firms are worse-off after new entry. Consider for instance a situation where entry costs are $14,440$. Then a single firm in the market would barely make a positive profit, and potential entrants might decide to stay out if they based their assessment on standard oligopoly settings (due to profits just covering entry costs). Yet, the market could actually accommodate a full 21 firms at the unique free entry equilibrium! □

The results that follow provide sufficient conditions that validate, for the highest equilibrium, the conventional wisdom that social welfare and consumer surplus increase with more competition, while industry profits decrease. We start the analysis starts with social welfare.

Theorem 17 *Assume A1-A5 are satisfied. Then, at the highest equilibrium, $W_{n+1} \geq W_n$ if at least one of the following conditions holds*

(i) $\int_0^{\overline{Z}_n} \left[P\left(t, \overline{Z}_{n+1}\right) - P\left(t, \overline{Z}_n\right)\right] dt \geq \overline{Z}_n \left[A\left(\overline{x}_{n+1}\right) - A\left(\overline{x}_n\right)\right]$*; or*

(ii) $\overline{x}_{n+1} \geq \overline{x}_n$.

In thinking about social and consumer welfare throughout, it is useful to keep in mind that since $P_2(Z, S) > 0$ by A1, and $\overline{Z}_{n+1} \geq \overline{Z}_n$ by Theorem 12 (i), the inverse demand shifts out as the number of firms increases from n to $n+1$, i.e. goes from $P\left(., \overline{Z}_n\right)$ to $P\left(., \overline{Z}_{n+1}\right)$. Hence, the area under the inverse demand changes through two effects: The shift in the demand curve and the change in equilibrium output. It follows that the left hand side of condition (i) is always positive. So this theorem identifies two sufficient conditions for welfare to increase: either one has diseconomies of scale ($A(.)$ is increasing) and decreasing per-firm output, or per-firm output being increasing in n. Network effects play a key role in giving rise to these two conditions. First, they give rise to the demand shift and to an increase in total output, which makes condition (i) more likely to hold. As seen earlier, they also weaken the business-stealing effect, thereby easing the conditions under which per-firm output increases in n.

We next study consumer surplus, for which our results differ markedly from their counterparts for the standard Cournot oligopoly.

Theorem 18 *Assume A1-A5 are satisfied. Then, at the highest equilibrium, $CS_{n+1} \geq CS_n$ if either (i) $P\left(\overline{Z}_{n+1}, \overline{Z}_{n+1}\right) \leq P\left(\overline{Z}_n, \overline{Z}_n\right)$; or (ii) $P_{12}(Z, S) \leq 0$.*

As a consequence of the so-called property of quasi-competitiveness, which under similar conditions holds in the standard Cournot game, condition (i) is always satisfied without network effects. Example 4, at the end of the section, shows the opposite sometimes happens in network industries. Katz and Shapiro (1985) clearly explain why this surprising result might occur here: If the marginal consumer has a strong network externality, then the increment in the expected network size generated by the larger number of firms in the market, will increase his/her willingness to pay for the product above that of the average consumer. As a consequence, the firms will be able to raise the price by more than the increase in the average consumer's willingness to pay for the product and consumer surplus will fall.

Our last theorem provides respective sufficient conditions for industry profits to increase or decrease with entry in a way that links the outcome with the price effect.

Theorem 19 *Assume A1-A5 are satisfied. Then, at the highest equilibrium,*

(i) $(n+1)\,\pi_{n+1} \geq n\pi_n$ *if* $P\left(\overline{Z}_{n+1}, \overline{Z}_{n+1}\right) - A\left(\overline{x}_{n+1}\right) \geq P\left(\overline{Z}_n, \overline{Z}_n\right) - A\left(\overline{x}_n\right)$ *and/or the conditions of Theorem 15 (i) are satisfied; and*

(ii) $(n+1)\,\pi_{n+1} \leq n\pi_n$ *if* $P\left(\overline{Z}_{n+1}, \overline{Z}_{n+1}\right) - P\left(\overline{Z}_{n+1}, \overline{Z}_n\right) \leq A\left(\overline{x}_{n+1}\right) - A\left(\frac{n+1}{n}\overline{x}_{n+1}\right)$.

Since the highest equilibrium industry output increases in n, part (i) is the trivial observation that, if mark-up increases in n, industry profits increase. The second statement is a corollary of Theorem 15 (i). The justification of Theorem 19 (ii) is quite similar to the previous one.

The final result (proof omitted) is a direct implication of previous ones, dealing with the important and widely used special case of constant returns to scale.

Corollary 20 *In addition to A1-A5, assume production costs are linear, i.e.* $C(x) = cx$ *with* $c \geq 0$. *Then, at the highest equilibrium, as the number of firms increases, (i) social welfare increases, and (ii) industry profit increases if market price increases.*

Example 4 illustrates how exogenous entry can affect consumer surplus and industry profits.

Example 4. Consider a Cournot oligopoly with no production costs and

$$P(Z,S) = \max\{a - Z/S^3, 0\} \text{ with } Z, S \in [0, nK] \text{ and } a, K > 1.$$

The reaction function of any given firm is

$$x(y,S) = \begin{cases} \max\left\{\left(aS^3 - y\right)/2, 0\right\} & \text{if } \left(aS^3 - y\right)/2 < K \\ K & \text{if } \left(aS^3 - y\right)/2 \geq K \end{cases}.$$

Thus, we have three possible FECE aggregate outputs: $Z_n = \left\{0, \sqrt{(n+1)/(na)}, nK\right\}$.

From a simple computation, consumer surplus is zero at the smallest equilibrium and, $CS_n = 1/(2nK)$ assuming $a \geq 1/(nK)^2$, at the highest one. Hence consumer surplus decreases in n for the highest equilibrium. This result is possible as Conditions (i) and (ii) in Theorem 18 are violated, i.e. the market price at the highest equilibrium increases in n and $P_{12}(Z,S) = 3/S^4 > 0$ for all Z, S.

Aggregate profit at the highest equilibrium is $n\pi_n = nK\left[a - 1/(nK)^2\right]$, which increases in n, as does the corresponding social welfare $W_n = anK - 1/(2nK)$. □

6 Conclusion

Building on basic insights from the theory of supermodular games, this paper has provided a thorough analysis of a standard static symmetric model of oligopolistic competition with nonlinear network effects. A minimal monotonicity structure on the model leads to industry output increasing in rivals' output and in the expected network size, thus yielding in one broad stroke existence of symmetric equilibrium as well as some key characterization results with a comparative statics flavor. In particular, industry viability, a known concept for which we provide novel theoretical foundations here, is shown to be enhanced by increases in the number of competitors as well as by technological progress. Likewise, respective sufficient conditions are derived for each dimension of market performance to increase or decrease with more competition. Due to the presence of demand-side increasing returns, the tendency for counterintuitive effects, which is fully characterized, is seen to be much stronger than in regular markets, where it is often due to scale economies. Most notably, price and per-firm profit can both increase with the number of firms, with the latter effect having no counterpart in regular markets even under scale economies (Amir and Lambson, 2000). Several illustrative examples with closed-form solutions are constructed, including one that captures exactly the well-known three-equilibria configuration that is broadly thought to capture expectations dynamics in telecommunication industries. In terms of policy implications, by identifying precise conditions for various possible effects to hold, our results provide solid theoretical foundations for some well-known policy prescriptions that need revisiting for network markets (Shapiro, 1998).

7 Proofs

This section provides the proofs for all the results of the paper, and also contains the statements and proofs of some useful intermediate results not given in the body of the paper.

The proof of Lemma 1 calls for an intermediate result.

Lemma 21 *Assume A1-A5 hold. Then* $\widetilde{\pi}(Z, y, S)$ *has the strict single-crossing property in* $(Z; S)$.

Proof of Lemma 21

To prove this result, first note that $\Delta_2(Z, S) > 0$ if and only if $\partial^2 \log P(Z, S) / \partial Z \partial S > 0$. We show that this condition implies that $\widetilde{\pi}(Z, y, S)$ has the strict single-crossing property in $(Z; S)$, i.e.

that for any $Z > Z'$ and $S > S'$,

$$\widetilde{\pi}\left(Z, y, S'\right) \geq \widetilde{\pi}\left(Z', y, S'\right) \Longrightarrow \widetilde{\pi}\left(Z, y, S\right) > \widetilde{\pi}\left(Z', y, S\right). \tag{10}$$

Since $\partial^2 \log P(Z, S) / \partial Z \partial S > 0$, $\log P(Z, S) - \log P(Z', S) > \log P(Z, S') - \log P(Z', S')$, or

$$\frac{P(Z, S)}{P(Z', S)} > \frac{P(Z, S')}{P(Z', S')}. \tag{11}$$

The left hand side of (10) can be rewritten as

$$(Z - y) P\left(Z, S'\right) - C(Z - y) \geq \left(Z' - y\right) P\left(Z', S'\right) - C\left(Z' - y\right). \tag{12}$$

Combining (11) and (12), we get

$$(Z - y) P(Z, S) \frac{P(Z', S')}{P(Z', S)} - C(Z - y) > \left(Z' - y\right) P\left(Z', S'\right) - C\left(Z' - y\right). \tag{13}$$

Multiplying both sides of (13) by $P(Z', S) / P(Z', S')$ we obtain

$$(Z - y) P(Z, S) - \frac{P(Z', S)}{P(Z', S')} C(Z - y) > \left(Z' - y\right) P\left(Z', S\right) - \frac{P(Z', S)}{P(Z', S')} C\left(Z' - y\right). \tag{14}$$

By A1, $P(Z', S) / P(Z', S') > 1$ and, by A2, $C(Z - y) \geq C(Z' - y)$. Thus, (14) implies

$$(Z - y) P(Z, S) - C(Z - y) > \left(Z' - y\right) P\left(Z', S\right) - C\left(Z' - y\right), \tag{15}$$

which is just the right hand side of (10). Hence, (10) holds. □

Proof of Lemma 1

Since $\partial^2 \widetilde{\pi}(Z, y, S) / \partial Z \partial y = \Delta_1(Z, y) > 0$, by A4, the maximand in (2) has strictly increasing differences in (Z, y). Furthermore, the feasible correspondence $(y, S) \longrightarrow [y, y + K]$ is ascending in y. Then, by Topkis's theorem [Theorem A.1, Appendix], every selection from the argmax of $\widetilde{\pi}(Z, y, S)$, $\widetilde{Z}(y, S)$, increases in y.

By Lemma 21, $\widetilde{\pi}(Z, y, S)$ has the strict single-crossing property in $(Z; S)$. In addition, the feasible correspondence $(y, S) \longrightarrow [y, y + K]$ does not depend on S. Then, by [Theorem A.2, Appendix] due to Milgrom and Shannon (1994), every selection from the argmax of $\widetilde{\pi}(Z, y, S)$, $\widetilde{Z}(y, S)$, is also increasing in S. □

Proof of Theorem 2

The following mapping, which can be thought of as a normalized cumulative best-response, is the key element in dealing with symmetric equilibria for any n[22]

$$\begin{aligned} B_n : [0, (n-1)K] \times [0, nK] &\longrightarrow 2^{[0,(n-1)K]\times[0,nK]} \\ (y, S) &\longrightarrow \left[\frac{n-1}{n}\left(x' + y\right), x' + y\right] \end{aligned}$$

where x' denotes a best-response output level by a firm to a joint output y by the other $(n-1)$ firms, given S. It is readily verified that the (set-valued) range of B_n is as given, i.e. if $x' \in [0, K]$ and $y \in [0, (n-1)K]$, then $((n-1)/n)(x' + y) \in [0, (n-1)K]$ and $x' + y \in [0, nK]$. Also, a fixed point of B_n is easily seen as a symmetric equilibrium, for it must satisfy both $\widehat{y} = ((n-1)/n)(\widehat{x}' + \widehat{y})$, or $\widehat{x}' = \widehat{y}/(n-1)$, and $\widehat{S} = \widehat{x}' + \widehat{y}$, which says that the responding firm produces as much as each of the other $(n-1)$ firms and the expected size of the network is fulfilled at equilibrium.

By Lemma 1 we know that every selection of $\widetilde{Z}(y, S)$ increases in y and S. Hence, for any fixed $n \in N$, every selection of B_n increases in (y, S), so that by Tarski's fixed point theorem [Theorem A.3, Appendix], it has a fixed point. As argued before, a fixed point of B_n is a symmetric equilibrium. This proves the first statement of Theorem 2.

To show that no asymmetric equilibria exists, it suffices to show that the correspondence $\widetilde{Z}(y, S)$ is strictly increasing (in the sense that all its selections are strictly increasing) in y, for each S. Thus, for all possible S, to each $Z' \in \widetilde{Z}(y, S)$ corresponds (at most) one y such that $Z' = x' + y$ with Z' being a best-response to y and S. In other words, for each equilibrium output Z', each firm must be producing the same $x' = Z' - y$, with $y = (n-1)x'$.

A4 implies that $\partial\widetilde{\pi}(Z, y, S)/\partial Z$ is strictly increasing in y, a property slightly stronger than strictly increasing differences in (Z, y). By Topkis (1998), Theorem 2.8.5 on p. 79, this property implies that $\widetilde{Z}(y, S)$ is strictly increasing in y for each S, whenever $\widetilde{Z}(y, S)$ is interior.[23] The second statement in Theorem 2 follows because, as argued in the previous paragraph, this condition guarantees no asymmetric equilibria exist. □

Proof of Lemma 3

The proof of this lemma follows directly from Amir and Lambson (2000), thus we omit it. □

Proof of Lemma 4

[22] See Amir and Lambson (2000) and Kwon (2007).

[23] This result was proved in Amir (1996) and Edlin and Shannon (1998).

By definition, an industry output of 0 is a FECE if $0 \in \widetilde{x}(0,0)$. This holds if and only if

$$\begin{aligned} \pi(0,0,0) &\geq \pi(x,0,0) \\ 0 &\geq xP(x,0) - C(x) \end{aligned}$$

for all $x \in [0,K]$. This proves our first statement. The second one follows because all the steps are independent of the number of firms in the market. □

The proof of Theorem 5 calls for several intermediate results, which will turn out to be useful for some other proofs as well. We first state sufficient conditions under which an increasing selection of $z_n(S)$ is differentiable for almost all S, and give a specific functional form for its slope. We then show that when 0 is part of the equilibrium set, then $z_n(0)$ is single-valued and right-continuous.

Lemma 22 *Assume A1-A5 are satisfied. Let $\widehat{z}_n$ be an increasing selection of $z_n(S)$, such that $\widehat{z}_n(S) \in (0,nK)$. Then $\widehat{z}_n(S)$ is differentiable for almost all S, and its slope is given by*

$$\frac{\partial \widehat{z}_n(S)}{\partial S} = \frac{-n\{P_1(\widehat{z}_n,S)P_2(\widehat{z}_n,S) - [P(\widehat{z}_n,S) - C'(\widehat{z}_n/n)]P_{12}(\widehat{z}_n,S)\}}{(n+1)[P_1(\widehat{z}_n,S)]^2 - n[P(\widehat{z}_n,S) - C'(\widehat{z}_n/n)]P_{11}(\widehat{z}_n,S) - P_1(\widehat{z}_n,S)C''(\widehat{z}_n/n)} \tag{16}$$

where $\widehat{z}_n$ stands for $\widehat{z}_n(S)$.

Proof of Lemma 22

If $\widehat{z}_n(S)$ is interior, it must satisfy the first order condition

$$P(\widehat{z}_n,S) + (\widehat{z}_n/n)P_1(\widehat{z}_n,S) - C'(\widehat{z}_n/n) = 0 \tag{17}$$

where $\widehat{z}_n$ stands for $\widehat{z}_n(S)$. Multiplying both sides of (17) by n

$$nP(\widehat{z}_n,S) + z_nP_1(\widehat{z}_n,S) - nC'(\widehat{z}_n/n) = 0. \tag{18}$$

Since $\widehat{z}_n(S)$ is increasing, it is differentiable almost everywhere (w.r.t. Lebesgue measure) and

$$\frac{\partial \widehat{z}_n(S)}{\partial S} = \frac{-[nP_2(\widehat{z}_n,S) + \widetilde{z}_nP_{12}(\widehat{z}_n,S)]}{(n+1)P_1(\widehat{z}_n,S) + \widehat{z}_nP_{11}(\widehat{z}_n,S) - C''(\widehat{z}_n/n)}. \tag{19}$$

Substituting $\widehat{z}_n(S)$ by its implicit value in (17), and multiplying the numerator and the denominator by $P_1(z_n,S)$, we obtain (16). □

Proof of Lemma 6

We first show that if $0 \in z_n(0)$, then $0 = z_n(0)$, i.e., $z_n(0)$ is a singleton. By Lemma 4 we know that $0 \in z_n(0)$ if and only if

$$xP(x,0) \leq C(x) \text{ for all } x \in [0,K]. \tag{20}$$

Since $P_1(Z,S) < 0$ by A1, it follows from (20) that $xP(x+y,0) < C(x)$ for all $x \in (0,K]$ and all $y > 0$. Hence, 0 is a dominant strategy in the standard Cournot game given $S = 0$. This proves that $z_n(0)$ is single-valued.

Since $P(0,0) = C'(0)$, the trivial outcome is an interior equilibrium. To show (4), take any sequence $S_k \downarrow 0$ such that $\widehat{z}_n$ is differentiable at S_k for all k (this is possible since the set of points of differentiability of an increasing function forms a dense subset of its domain). Since $\widehat{z}_n$ is increasing, it has left and right limits at every point, so $\lim_{k\to\infty} \widehat{z}_n(S_k)$ exists. Since $z_n(.)$ is u.h.c., $\lim_{k\to\infty} \widehat{z}_n(S_k) \in z_n(0) = \{0\}$, so that by the earlier part of this proof, $\lim_{k\to\infty} \widehat{z}_n(S_k) = 0$.

Now consider (19) with $S = S_k$. By Assumption A1 and the fact that $\lim_{k\to\infty} \widehat{z}_n(S_k) = 0$, the right-hand side of (19) is right-continuous in S at 0. Taking limits as $k \longrightarrow \infty$, (4) follows. Since this argument is clearly independent of the particular (increasing) selection $\widehat{z}_n$ and of the sequence (S_k) chosen, $\partial z_n(S)/\partial S|_{S=0}$ is single-valued, continuous at 0, and given by (4).

The fact that $z_n'(0) = 0$ if the trivial equilibrium in not interior follows directly from our previous arguments, thus we omit this proof. □

We next show that, for all $S \in [0, nK]$, any argmax of a fictitious objective function $\Pi(Z,S)$ is an element of $z_n(S)$.

Lemma 23 *Assume A1-A5 are satisfied and $C(.)$ is convex. Given any $n \in N$ and $S \in [0,nK]$, if $Z' \in \arg\max\{\Pi(Z,S) : 0 \leq Z \leq nK\}$ then $Z' \in z_n(S)$.*

Proof of Lemma 23

We show that if Z^* is an argmax of $\Pi(Z,S)$, then Z^* is the industry output of a symmetric Cournot equilibrium with exogenous S. Let $Z^* = x^* + y^*$, with $x^* = Z^*/n$ and $y^* = (n-1)x^*$, and consider $Z' = x' + y^*$, with $x' \in [0,K]$. Then x' denotes a possible deviation of a given firm from its equilibrium output x^*. We next show this unilateral deviation is never profitable.

Since Z^* is a maximizer of $\Pi(Z,S)$, then $\Pi(Z^*,S) \geq \Pi(Z',S)$, which is equivalent to say

$$\frac{n-1}{n}\int_0^{x^*+y^*} P(t,S)\,dt + x^*P(x^*+y^*,S) - nC(x^*) \geq$$
$$\frac{(n-1)}{n}\int_0^{x'+y^*} P(t,S)\,dt + \frac{(x'+y^*)}{n}P\left(x'+y^*,S\right) - nC\left(\frac{x'+y^*}{n}\right) \tag{21}$$

Then we have

$$\begin{aligned}
& x^*P(x^*+y^*,S) - C(x^*) \\
\geq\ & \frac{n-1}{n}\int_0^{x'+y^*} P(t,S)\,dt + \frac{(x'+y^*)}{n}P\left(x'+y^*,S\right) - nC\left(\frac{x'+y^*}{n}\right) \\
& -\frac{n-1}{n}\int_0^{x^*+y^*} P(t,S)\,dt + (n-1)\,C(x^*) \\
\geq\ & \frac{n-1}{n}\int_{x^*+y^*}^{x'+y^*} P(t,S)\,dt + \frac{(x'+y^*)}{n}P\left(x'+y^*,S\right) - C\left(x'\right) \\
\geq\ & \frac{(n-1)(x'-x^*)}{n}P\left(x'+y^*,S\right) + \frac{(x'+y^*)}{n}P\left(x'+y^*,S\right) - C\left(x'\right) \\
=\ & x'P\left(x'+y^*,S\right) - C\left(x'\right).
\end{aligned}$$

The first inequality follows from (21), after rearranging terms. The second one holds as we assumed $C(.)$ is convex (and $y^* = (n-1)\,x^*$), and the last one by A1, $P_1(Z,S) < 0$. Since x' is arbitrary, this argument shows that x^* is a symmetric Cournot equilibrium. □

Proof of Theorem 5

Part (i) holds because, if the trivial outcome (zero output) is not part of the equilibrium set, Theorem 2 guarantees there is a FECE with strictly positive industry output.

Parts (ii) and (iii) are both based on the following argument. By Lemma 3, the maximal and minimal selections of $z_n(S)$, $\overline{z}_n(S)$ and $\underline{z}_n(S)$, increase in S. Assume, for the moment, there exists an $S' \in (0,nK]$ such that $\overline{z}_n(S') \geq S'$. If we restrict attention to the values of S in $[S',nK]$, it follows that $\overline{z}_n(S) \in [S',nK]$ because $\overline{z}_n(.)$ is increasing and $\overline{z}_n(S') \geq S'$. Therefore, for all $S \in [S',nK]$, $\overline{z}_n(S)$ is an increasing function that maps $[S',nK]$ into itself. Hence, by Tarski's fixed point theorem [Theorem A.3, Appendix], there is an $S' \leq S'' \leq nK$ such that $\overline{z}_n(S'') = S''$. Since this condition implies $\overline{z}_n(S'')$ is a strictly positive FECE, the existence of a nontrivial equilibrium reduces to showing there is at least one $S \in (0,nK]$ for which $\overline{z}_n(S) \geq S$.

To prove Part (ii), we show $z_n'(0) > 1$. By Lemma 6, $z_n'(0) > 1$ if, given $P_1(0,0)+P_2(0,0) > 0$,

$$n > \left[-P_1(0,0) + C''(0)\right] / \left[P_1(0,0) + P_2(0,0)\right].$$

Then the existence of a nontrivial FECE follows by the argument in the previous paragraph, as Lemma 6 and the property $z_n'(0) > 1$, imply there exists a small $\varepsilon > 0$ for which $\overline{z}_n(\varepsilon) > \varepsilon$. This completes the proof of Part (ii).

The inequality in Part (iii) is equivalent to say there is some $S \in (0, nK]$ and some $Z' \geq S$ for which $n[\Pi(Z', S) - \Pi(Z, S)] \geq 0$ for all $Z \leq S$. As a consequence, the largest argmax of $\Pi(Z, S)$ must be larger than S. Call this argmax $\widehat{Z}$. Our proof follows because $\widehat{Z} \in z_n(S)$, by Lemma 23, and this ensures there is an $S \in (0, nK]$ for which an element of $z_n(S)$ is higher than S. $\square$

Proof of Theorem 11

We will show Part (i) for a change in n. The proof for α is almost identical, so we omit it. [Amir and Lambson (2000), Theorem 2.2 (b)] shows that $\overline{z}_n(S)$ and $\underline{z}_n(S)$ are increasing in n. Given this, the claim follows directly from Definition 10.

To prove Part (ii), observe that if the trivial equilibrium holds and $P_1(0,0) + P_2(0,0) \leq 0$, then, by Lemma 6, $\overline{z}_n'(0) < 1 \; \forall \; n$, so that 0 is a stable equilibrium $\forall \; n$. This ends our proof. $\square$

Proof of Theorem 12

The maximal and minimal selections of B_n (as defined in the proof of Theorem 2) denoted, respectively, $\overline{B}_n$ and $\underline{B}_n$, exist by Topkis's theorem. Furthermore, the largest equilibrium values of y_n and Z_n , $(\overline{y}_n, \overline{Z}_n)$, constitute the largest fixed point of $\overline{B}_n$. Since $(n-1)/n$ is increasing in n, $\overline{B}_n$ is increasing in n for all (y, S). Since $\overline{B}_n$ is also increasing in both y and S, the largest fixed point of $\overline{B}_n$, $(\overline{y}_n, \overline{Z}_n)$, is also increasing in n (see Milgrom and Roberts, 1990). A similar argument, using the selection $\underline{B}_n$, establishes that $\left(\underline{y}_n, \underline{Z}_n\right)$ increases in n as well. This shows part (i).

Part (ii) follows directly from Part (i) since $dP(Z,Z)/dz = P_1(Z,Z) + P_2(Z,Z)$. $\square$

Proof of Theorem 13

Consider the following mapping

$$\begin{aligned} M_n : [0, nK] \quad &\longrightarrow \quad 2^{[0,K]} \\ Z \quad &\longrightarrow \quad \widetilde{x} = \left\{x : P(Z,Z) + xP_1(Z,Z) - C'(x) = 0\right\}. \end{aligned} \tag{22}$$

Then M_n maps industry output into the solution of a fictitious first order condition, which coincides with that of an interior FECE when $x = Z/n$ and $Z = Z_n$.

Totally differentiating this first order condition with respect to n, we have

$$\{P_1(Z,Z) + P_2(Z,Z) + \widetilde{x}\,[P_{11}(Z,Z) + P_{12}(Z,Z)]\}\,\frac{dz}{dn} = 0. \tag{23}$$

Substituting in (23) $\widetilde{x}$ by $[C'(Z/n) - P(Z,Z)]/P_1(Z,Z)$, and rearranging terms, we get

$$-\frac{1}{P_1(Z,Z)}\left\{\left[P(Z,Z) - C'(Z/n)\right][P_{11}(Z,Z) + P_{12}(Z,Z)] - P_1(Z,Z)\,[P_1(Z,Z) + P_2(Z,Z)]\right\}\frac{dz}{dn} = 0. \tag{24}$$

Substituting $g(Z)$ from (8) into (24), we get

$$-\frac{1}{P_1(Z,Z)}g(Z)\,\frac{dz}{dn} = 0. \tag{25}$$

By A1, $P_1(Z,Z) < 0$. Also, by Theorem 12 (i), the extremal equilibrium industry outputs increase in n. Then, if $g(Z) \geq (\leq)\,0$ over $\left[\overline{Z}_n, \overline{Z}_{n+1}\right]$, the mapping M_n increases (decreases) in n at the largest equilibrium industry output. Theorem 13 follows because if M_n increases (decreases) in n at the largest equilibrium industry output, then $\overline{x}_n$ also increases (decreases) with this parameter. By a similar argument it can be shown that this is also true for $\underline{x}_n$. □

Proof of Corollary 14

Inequality 9 equals function $g(Z)$ when the firms face no cost of production. Then the first claim follows directly from Theorem 13 (i).

The first term in the left hand side of (9) is always positive by A5. As the log-convexity of $P(Z,S)$ in Z guarantees the second term is also positive, this is a sufficient condition for the required inequality. □

Proof of Theorem 15

Consider the following inequalities

$$\begin{aligned}
\pi_{n+1} &= \overline{x}_{n+1}P\left(\overline{x}_{n+1} + \overline{y}_{n+1}, \overline{Z}_{n+1}\right) - C(\overline{x}_{n+1}) \\
&\geq \overline{x}_n P\left(\overline{x}_n + \overline{y}_{n+1}, \overline{Z}_{n+1}\right) - C(\overline{x}_n) \\
&\geq \overline{x}_n P\left(\overline{x}_{n+1} + \overline{y}_{n+1}, \overline{Z}_{n+1}\right) - C(\overline{x}_n) \\
&\geq \overline{x}_n P\left(\overline{x}_n + \overline{y}_n, \overline{Z}_n\right) - C(\overline{x}_n) \\
&= \pi_n.
\end{aligned}$$

The first inequality follows by the Cournot equilibrium property. The second one is from $\overline{x}_{n+1} \geq \overline{x}_n$ and A1. (The fact that $\overline{x}_{n+1} \geq \overline{x}_n$ here follows by Theorem 13 (i) because we assumed all its required conditions are satisfied.) The third inequality follows because our assumptions imply $P\left(\overline{Z}_{n+1},\overline{Z}_{n+1}\right)\geq P\left(\overline{Z}_{n},\overline{Z}_{n}\right)$. Therefore, $\overline{\pi}_{n+1}\geq\overline{\pi}_{n}$. By a similar argument it can be shown that this is also true for the equilibrium per-firm profits evaluated at the minimal equilibrium outputs. This shows Part (i). We omit the proof of Part (ii) as it is almost identical to the last one. □

Proof of Corollary 16

If $P_{11}(Z,Z)+P_{12}(Z,Z)=0$, then $g(Z)=-P_{1}(Z,Z)\left[P_{1}(Z,Z)+P_{2}(Z,Z)\right]$. By A1, $P_{1}(Z,Z)<0$. Then the sign of $g(Z)$ is equal to the sign of $P_{1}(Z,Z)+P_{2}(Z,Z)$, and Corollary 16 (i) and (ii) follow by Theorems 13 (i) and 15 (i), respectively. □

Proof of Theorem 17

To show Part (i) consider

$$\begin{aligned} W_{n+1}-W_{n} &= \int_{0}^{\overline{Z}_{n+1}}P\left(t,\overline{Z}_{n+1}\right)dt-\overline{Z}_{n+1}A\left(\overline{x}_{n+1}\right)-\left[\int_{0}^{\overline{Z}_{n}}P\left(t,\overline{Z}_{n}\right)dt-\overline{Z}_{n}A\left(\overline{x}_{n}\right)\right] \\ &\geq \int_{0}^{\overline{Z}_{n}}P\left(t,\overline{Z}_{n+1}\right)dt-\overline{Z}_{n}A\left(\overline{x}_{n+1}\right)-\left[\int_{0}^{\overline{Z}_{n}}P\left(t,\overline{Z}_{n}\right)dt-\overline{Z}_{n}A\left(\overline{x}_{n}\right)\right] \\ &\geq 0. \end{aligned}$$

The first inequality follows because $P\left(t,\overline{Z}_{n+1}\right)-A\left(\overline{x}_{n+1}\right)\geq 0$ for all $t\leq\overline{Z}_{n+1}$, and $\overline{Z}_{n+1}\geq\overline{Z}_{n}$ by Theorem 12 (i). The second inequality holds by the assumed conditions.

To show Part (ii) let us define $V_{n}(x,S)=\int_{0}^{nx}P(t,S)\,dt-nC(x)$. Notice $V_{n}(x,S)$ is concave in x since $n\left[nP_{1}(nx,S)-C''(x)\right]<0$ by both A1 and A4. In addition,

$$\begin{aligned} \int_{0}^{\overline{Z}_{n+1}}P\left(t,\overline{Z}_{n+1}\right)dt &= \int_{0}^{n\overline{x}_{n+1}}P\left(t,\overline{Z}_{n+1}\right)dt+\int_{n\overline{x}_{n+1}}^{\overline{Z}_{n+1}}P\left(t,\overline{Z}_{n+1}\right)dt \\ &\geq \int_{0}^{n\overline{x}_{n+1}}P\left(t,\overline{Z}_{n+1}\right)dt+\overline{x}_{n+1}P\left(\overline{Z}_{n+1},\overline{Z}_{n+1}\right) \end{aligned} \tag{26}$$

where the inequality follows by A1. The following steps show our result

$$\begin{aligned}
W_{n+1} - W_n &= \int_0^{(n+1)\overline{x}_{n+1}} P\left(t, \overline{Z}_{n+1}\right) dt - (n+1)\, C\left(\overline{x}_{n+1}\right) - \left[\int_0^{n\overline{x}_n} P\left(t, \overline{Z}_n\right) dt - nC\left(\overline{x}_n\right)\right] \\
&\geq \pi_{n+1} + \int_0^{n\overline{x}_{n+1}} P\left(t, \overline{Z}_{n+1}\right) dt - nC\left(\overline{x}_{n+1}\right) - \left[\int_0^{n\overline{x}_n} P\left(t, \overline{Z}_n\right) dt - nC\left(\overline{x}_n\right)\right] \\
&\geq \pi_{n+1} + \int_0^{n\overline{x}_{n+1}} P\left(t, \overline{Z}_{n+1}\right) dt - nC\left(\overline{x}_{n+1}\right) - \left[\int_0^{n\overline{x}_n} P\left(t, \overline{Z}_{n+1}\right) dt - nC\left(\overline{x}_n\right)\right] \\
&= \pi_{n+1} + V_n\left(\overline{x}_{n+1}, \overline{Z}_{n+1}\right) - V_n\left(\overline{x}_n, \overline{Z}_{n+1}\right) \\
&\geq \pi_{n+1} + \left[\partial V_n\left(\overline{x}_{n+1}, \overline{Z}_{n+1}\right) / \partial x\right] \left(\overline{x}_{n+1} - \overline{x}_n\right) \\
&= \pi_{n+1} + n\left[P\left(n\overline{x}_{n+1}, \overline{Z}_{n+1}\right) - C'\left(\overline{x}_{n+1}\right)\right] \left(\overline{x}_{n+1} - \overline{x}_n\right) \\
&\geq \pi_{n+1} + n\left[P\left((n+1)\,\overline{x}_{n+1}, \overline{Z}_{n+1}\right) - C'\left(\overline{x}_{n+1}\right)\right] \left(\overline{x}_{n+1} - \overline{x}_n\right) \\
&\geq 0.
\end{aligned}$$

The first inequality follows from inequality (26), the second one by A1 and Theorem 12 (i) and the third one by the concavity of $V_n(x, S)$ in x. The fourth inequality holds by A1 and because we assumed $\overline{x}_{n+1} \geq \overline{x}_n$, and the last one by the Cournot property. This completes our proof. □

Proof of Theorem 18

The proof of Part (i) follows directly from Theorem 12 (i).

The following steps prove Part (ii)

$$\begin{aligned}
CS_{n+1} - CS_n &= \int_0^{\overline{Z}_{n+1}} \left[P\left(t, \overline{Z}_{n+1}\right) - P\left(\overline{Z}_{n+1}, \overline{Z}_{n+1}\right)\right] dt - \int_0^{\overline{Z}_n} \left[P\left(t, \overline{Z}_n\right) - P\left(\overline{Z}_n, \overline{Z}_n\right)\right] dt \\
&\geq \int_0^{\overline{Z}_n} \left[P\left(t, \overline{Z}_{n+1}\right) - P\left(\overline{Z}_{n+1}, \overline{Z}_{n+1}\right)\right] dt - \int_0^{\overline{Z}_n} \left[P\left(t, \overline{Z}_n\right) - P\left(\overline{Z}_n, \overline{Z}_n\right)\right] dt \\
&= \overline{Z}_n \left[P\left(\overline{Z}_n, \overline{Z}_n\right) - P\left(\overline{Z}_{n+1}, Z_n\right)\right] \\
&\quad - \int_0^{\overline{Z}_n} \left\{\left[P\left(\overline{Z}_{n+1}, \overline{Z}_{n+1}\right) - P\left(\overline{Z}_{n+1}, \overline{Z}_n\right)\right] - \left[P\left(t, \overline{Z}_{n+1}\right) - P\left(t, \overline{Z}_n\right)\right]\right\} dt \\
&\geq \overline{Z}_n \left[P\left(\overline{Z}_n, \overline{Z}_n\right) - P\left(\overline{Z}_{n+1}, Z_n\right)\right] \\
&\geq 0.
\end{aligned}$$

The first inequality follows directly from $P_1(Z, S) < 0$ and Theorem 12 (i). The next step is obtained from the previous one by adding and subtracting $\int_0^{\overline{Z}_n} P\left(Z_{n+1}, \overline{Z}_n\right) dt$, and rearranging terms. To justify the second inequality notice that $P_{12}(Z, S) \leq 0$ is sufficient for

$$\int_0^{\overline{Z}_n} \left[P\left(t, \overline{Z}_{n+1}\right) - P\left(t, \overline{Z}_n\right)\right] dt \geq \int_0^{\overline{Z}_n} \left[P\left(\overline{Z}_{n+1}, \overline{Z}_{n+1}\right) - P\left(\overline{Z}_{n+1}, \overline{Z}_n\right)\right] dt.$$

Our last step is true since $P_1(Z,S) < 0$.

Hence, $P_{12}(Z,S) \leq 0 \; \forall \; Z, S \in [0, nK]$ is sufficient for $CS_{n+1} - CS_n \geq 0$, or $CS_{n+1} \geq CS_n$. □

Proof of Theorem 19

To show Part (i), for an extremal equilibrium industry output, let us consider

$$\begin{aligned}(n+1)\pi_{n+1} - n\pi_n &= \overline{Z}_{n+1}\left[P\left(\overline{Z}_{n+1}, \overline{Z}_{n+1}\right) - A\left(\overline{x}_{n+1}\right)\right] - \overline{Z}_n\left[P\left(\overline{Z}_n, \overline{Z}_n\right) - A\left(\overline{x}_n\right)\right] \\ &\geq \overline{Z}_n\left[P\left(\overline{Z}_{n+1}, \overline{Z}_{n+1}\right) - A\left(\overline{x}_{n+1}\right)\right] - \overline{Z}_n\left[P\left(\overline{Z}_n, \overline{Z}_n\right) - A\left(\overline{x}_n\right)\right]\end{aligned}$$

Since $P\left(\overline{Z}_{n+1}, \overline{Z}_{n+1}\right) \geq A\left(\overline{x}_{n+1}\right)$, the inequality is due to Theorem 12 (i). The first part of Theorem 19 follows. The second statement in Part (i) follows from Theorem (15) (i). The proof of Part (ii) is similar, so we omit it. □

APPENDIX

In an attempt to make this paper self-contained, we provide a summary of all lattice-theoretic notions and results used here. Since this paper deals with real decision and parameter spaces, every theorem that follows is a special case of the original one (see Topkis, 1998).

A function F: $R_+^2 \to R$ is supermodular if, for $x_1 \geq x_2$, $y_1 \geq y_2$,

$$F(x_1, y_1) - F(x_2, y_1) \geq F(x_1, y_2) - F(x_2, y_2). \tag{27}$$

If F is twice continuously differentiable, Topkis's (1978) Characterization Theorem says that supermodularity is equivalent to $\dfrac{\partial^2 F}{\partial x \partial y} \geq 0$, for all x, y. Furthermore, $\dfrac{\partial^2 F}{\partial x \partial y} > 0$ implies that F is strictly supermodular, the latter notion being defined by a strictly inequality in (27).

F has the single-crossing property or SCP in $(x; y)$ if, for $x_1 \geq x_2$, $y_1 \geq y_2$,

$$F(x_1, y_2) - F(x_2, y_2) \geq 0 \Longrightarrow F(x_1, y_1) - F(x_2, y_1) \geq 0 \tag{28}$$

Note that (27) implies (28), while the converse is generally not true. Additionally, (27) is a cardinal notion while (28) is ordinal. Thus, the SCP is sometimes also referred to as ordinal supermodularity.

For $x \in R_+$, let $A(x) = [a_1(x), a_2(x)] \subset R_+$, with $a_1(.)$ and $a_2(.)$ being real-valued functions. We say $A(.)$ is ascending (in x) if a_1 and a_2 are increasing in x. The following results on monotone maximizers are central to our approach.

Theorem A.1. (Topkis, 1978). Assume that (i) F is upper-semi continuous (or u.s.c.) and supermodular in (x, y) and (ii) $A(.)$ is ascending. Then, the maximal and minimal selections of

$y^*(x) \equiv \arg\max_{y \in A(x)} F(x, y)$ are increasing functions. Furthermore, if F is strictly supermodular, then every selection of $y^*(.)$ is increasing.

Theorem A.2. (Milgrom and Shannon, 1994). Assume that (i) F is u.s.c. and has the SCP in $(x; y)$ and (ii) $A(.)$ is ascending. Then, the conclusion of Theorem A.1. holds.

Theorem A.3. Let $n \geq 1$ and $B : X_{i=1}^{n}[a_i, b_i] \rightarrow X_{i=1}^{n}[a_i, b_i]$ be an increasing function. Then B has a fixed point. (This theorem is a special case of Tarski's Fixed Point Theorem.)

Our equilibrium comparisons are based on the following result (Milgrom and Roberts, 1990).

Theorem A.4. Let $B_t : X_{i=1}^{n}[a_i, b_i] \rightarrow X_{i=1}^{n}[a_i, b_i]$ be an increasing function, $\forall t$, such that $B_t(x)$ is also increasing in t, $\forall x$. Then the minimal and maximal fixed-points of B_t increase in t.

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